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KILLED RANDOM PROCESSES AND HEAT KERNELS

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Let $V(x) \geq 0$ be a given function tending to a constant at infinity. It is well known that the density of the Brownian motion B_t killed at the infinitesimal rate V is a Green's function for the heat operator with such a potential. With an appropriate generalization, its Laplace transform also gives the density of $\int_0^t V(B_s)\,ds$. We construct such a Green's function via spectral analysis of the classical one-dimensional stationary Schrödinger operator.

Keywords: Brownian motion, heat equation propagator

1. Brownian motion and killing

In this introductory section, we recall several well-known aspects of the classical theory of the Brownian motion (BM) B_t (see [1] for more details). We are interested in certain aspects of the theory where the Green's function $G(t, x \,|\, t', x') \equiv G(t - t', x \,|\, x')$ for the heat operator with a negative "time-independent" potential, i.e.,

$$LG \equiv \big(-\partial_t + \partial_{xx} - V(x)\big) G(t, x \,|\, t', x') = -\delta(t - t')\delta(x - x'), \tag{1}$$

plays a crucial role. The construction of this propagator and its relation to the spectral analysis of the classical one-dimensional stationary Schrödinger operator is considered in Sec. 2. Assuming that $V(x)$ tends to a constant as $|x| \to \infty$, we show how to implement this construction. In Sec. 3, we give a concrete construction of $G(t, x \,|\, t', x')$ when $V(x)$ corresponds to the simplest reflectionless potential of the Schrödinger operator.

We recall that BM is a stochastic process B_t that models a random walk, i.e., it describes the erratic motion of a particle that can move to the right or left with equal probability at each instant. Here, $B_t \equiv B_t(\omega)$ represents the position at time t of the Brownian traveler. If motion starts at x': $B_0 = x'$ and is assumed to be isotropic and homogeneous in space and time, then B_t has the density given by the classical heat kernel

$$P\big(B_t \in [x, x + dx)\big) = \frac{1}{\sqrt{4\pi t}} \exp\left[-\frac{(x - x')^2}{4t}\right] dx.$$

Equation (1) arises as follows. In addition, we suppose that a random killing mechanism is introduced such that B_t "disappears" at a random time τ or, more precisely, attains a (new) death state ∂. We call the resulting process $\widehat{B}_t = \partial\theta(t - \tau) + B_t\theta(\tau - t)$ the BM with killing $\widehat{B}_t \in \mathbb{R} \cup \{\partial\}$ (or KBM). Let $\varphi(t) \equiv P(\widehat{B}_t \neq \partial)$ be the probability that $\widehat{B}_t$ survives up to time t. We suppose that given that $\widehat{B}_t = x \in \mathbb{R}$ ($\widehat{B}_t$ took a value x and hence has not yet been killed), the probability of being killed at any time $t + h > t$ is $o(h)$; concretely,

$$P(\widehat{B}_{t+h} \neq \partial \,|\, \widehat{B}_t = x) = 1 - V(x)h + o(h). \tag{2}$$

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Of course, $P(\widehat{B}_{t+h} \neq \partial \,|\, B_t = \partial) = 0$. Therefore, $V(x) \geq 0$ is the infinitesimal rate of killing of the Brownian particle. The former rules define the killing mechanism and by the total probability theorem imply that

$$P(\widehat{B}_t \neq \partial) = \exp\biggl\{-\int_0^t V(B_s)\,ds\biggr\}. \tag{3}$$

Indeed, we have

$$P(\widehat{B}_{t+h} \neq \partial) = P(\widehat{B}_{t+h} \neq \partial \,|\, \widehat{B}_t \neq \partial)P(\widehat{B}_t \neq \partial) + P(\widehat{B}_{t+h} \neq \partial \,|\, \widehat{B}_t = \partial)P(\widehat{B}_t = \partial).$$

Hence,

$$\varphi(t+h) = \varphi(t)\bigl(1 - V(B_t)h\bigr) + o(h),$$

and letting $h \to 0$, we obtain the differential equation with the initial condition

$$\frac{d\varphi}{dt} = -\varphi(t)V(B_t), \qquad \varphi(0) = 1,$$

and path integral (3) is recovered.

Given that $B_0 = x'$, KBM $\widehat{B}_t$ is then determined by giving the density

$$P\bigl(\widehat{B}_t \in [x, x+dx)\bigr) = P\bigl(B_t \in [x, x+dx), \tau > t\bigr) \equiv f(t, x \,|\, x')\,dx \tag{4}$$

and the distribution of the death time

$$P(\tau \leq t) = 1 - P(\widehat{B}_t \neq \partial) = 1 - \int_{\mathbb{R}} f(t, x \,|\, x')\,dx. \tag{5}$$

This density is recovered by the classical Feynman–Kac formula of probability and quantum mechanics establishing that the kernel $f(t, x \,|\, x')$ of path integral (3) is a solution of (1): $Lf(x, t \,|\, x') = \delta(t)\delta(x - x')$. In particular, if $V(x) = b^2$, then we have

$$f(x, t \,|\, x') = \frac{e^{-b^2 t - (x-x')^2/(4t)}}{\sqrt{4\pi t}}\theta(t), \qquad P(\tau \leq t) = 1 - e^{-b^2 t},$$

$$P(B_t \text{ is killed in finite time}) = \lim_{t\to\infty} P(\tau \leq t) = 1,$$

and the Brownian traveler is killed in a finite time with certainty.

These ideas find an interesting application in the problem of determining the density $\pi(t, z \,|\, x')dz \equiv P(Z_t \in dz)$ of the integrated process $Z_t \equiv \int_0^t V(B_s)\,ds$ given the initial values $B_0 = x'$ and $Z_0 = 0$. For this, given $V(x)$, we consider the family $\widetilde{V}(x; p) \equiv pV(x)$ of killing functions indexed by the positive parameter $p \geq 0$. Let $\widehat{B}_t^{p)}$ be the corresponding KBM and $f(t, x; p \,|\, x')$ be its density: $P\bigl(\widehat{B}_t^{p)} \in [x, x+dx)\bigr) \equiv f(t, x; p \,|\, x')\,dx$. The total probability theorem gives

$$P(\widehat{B}_t^{p)} \neq \partial) = \exp\biggl\{-p\int_0^t V(B_s)\,ds\biggr\} = \int_0^\infty e^{-pz}\pi(t, z \,|\, x')\,dz. \tag{6}$$

It follows from (5) and (6) that

$$\zeta(t; p \,|\, x') = \int f(t, x; p \,|\, x')\,dx = \int_0^\infty e^{-pz}\pi(t, z \,|\, x')\,dz.$$

Inverting the Laplace transform, we have

$$\pi(t, z \,|\, x') = \frac{1}{2\pi i} \int_\Gamma \zeta(t; p \,|\, x') e^{pz} \, dp, \tag{7}$$

where Γ is the classical Bromwich contour running along a line parallel to the imaginary axis that leaves all singularities of $\zeta(t; p \,|\, x')$ in the complex p plane to the left.

But complete information about the correlated pair (B_t, Z_t) requires its joint density $\Pi(t, x, z \,|\, x')$ defined by $\Pi(t, x, z \,|\, x') \, dx \, dz \equiv P\big(B_t \in [x, x + dx), Z_t \in [z, z + dz)\big)$. Similarly as above, we can prove that

$$\Pi(t, x, z \,|\, x') = \frac{1}{2\pi i} \int_\Gamma f(t, x; p \,|\, x') e^{pz} \, dp. \tag{8}$$

Again appealing to the Feynman–Kac formula, we find that $f(t, x; p \,|\, x')$ solves (1) with the potential $\widetilde{V}(x; p) \equiv pV(x)$:

$$\big(-\partial_t + \partial_{xx} - pV(x)\big) f(t, x; p \,|\, x') = -\delta(t - t')\delta(x - x').$$

The problem of determining the statistical distribution of $\int_0^t V(B_s) \, ds$ thus reduces to obtaining the density of the KBM with the potential $pV(x)$. As we now see, this is generally a difficult problem interwoven with classical spectral analysis for the one-dimensional stationary Schrödinger operator.

2. Determining the density of a killed BM and heat propagators

We now show how to determine the density of the KBM for a certain class of potentials. We assume that the function $V(x)$ satisfies $V(x) \geq 0$ and that $V(x) \equiv b^2 - u(x)$ where b is a certain constant and $u(x)$ satisfies

$$\lim_{|x| \to \infty} u(x) = 0, \qquad \int \big(1 + |x|\big) \big|u(x)\big| \, dx < \infty. \tag{9}$$

We find that the Green's function is constructed in terms of eigenfunctions of the one-dimensional Schrödinger operator $A(x, \partial_x) \equiv \partial_{xx} + k^2 + u(x)$, where $k \equiv k_R + ik_I \in \mathbb{C}$ is a complex parameter (the identification $b = k_I$ is used later). We follow [2], where these ideas are developed in the context of the classical Kadomtsev–Petviashvili equation

$$(u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0$$

(we note that some preliminary work in this regard also appeared in [3]). We first recall several basic facts about the spectral theory of the former operator (see [4], [5]) for more details).

Let $\phi_\pm(x, k)$ and $\psi_\pm(x, k)$ be eigenfunctions of the stationary Schrödinger operator,

$$A(x, \partial_x)\phi_\pm(x, k) = A(x, \partial_x)\psi_\pm(x, k) = 0, \tag{10}$$

satisfying the conditions

$$\phi_\pm(x, k) = e^{\mp ikx}, \quad x \to -\infty, \qquad \psi_\pm(x, k) = e^{\mp ikx}, \quad x \to \infty. \tag{11}$$

If $u(x)$ satisfies condition (9), then the former functions exist and are analytic functions of $k \equiv k_R + ik_I$ on $\mathbb{C}_\pm$ (the upper and lower k half-planes) with limits at the boundary $\{k_I = 0\}$ and related by

$$\phi_+(x, k) = a(k)\psi_-(x, k) + b(k)\psi_+(x, k), \quad k \in \mathbb{R}, \tag{12}$$

for certain functions $a(k)$ and $b(k)$ (see [5]), where $a(k)$ is an analytic function of k on $\mathbb{C}_+$ having a finite set $\{k_j \equiv i\kappa_j, \kappa_j \in \mathbb{R}^+\}_{j=1,\dots,N}$ of (simple) zeroes. It turns out that $\phi_+(x,k)$ and $\psi_+(x,k)$ are proportional at these points: $\phi_+(x,k_j) = \beta_j \psi_+(x,k_j)$, where β_j is some complex constant. This along with (11) implies that $\phi_+(x,k_j)$ and $\psi_+(x,k_j)$ decay exponentially. The reflection coefficient $\rho(k) \equiv b/a(k)$, the "norming" constants β_j, and the zeroes $\left\{k_j \equiv i\kappa_j, \kappa_j \in \mathbb{R}^+ : a(k_j) = 0\right\}_{j=1,\dots,N}$ are the continuous and discrete scattering data of the one-dimensional Schrödinger operator, and $\psi_j(x) \equiv \psi_+(x,k_j)$ are the eigenfunctions of the discrete spectrum.

Let the continuous and discrete parts of the Green's function be

$$G_{\mathrm{c}}(t,x\,|\,x') = \frac{\theta(t)}{2\pi} \lim_{L\to\infty} \int_{-L}^{L} e^{-t(l^2+2ik_I l)} g(x,x',l+ik_I)\,dl, \tag{13}$$

$$G_{\mathrm{d}}(t,x\,|\,x') = i \sum_{\kappa_j \geq k_I} e^{(-k_I^2+\kappa_j^2)t} g_j(x,x')\theta(-t), \tag{14}$$

where we define $g(x,x',k)$ on $\mathbb{C}_+$ as

$$g(x,x',k) \equiv \frac{\phi_+(x,k)\psi_+(x',k)}{a_+(k)}, \quad k_I > 0. \tag{15}$$

Finally, the Green's function is taken to be

$$G(t,x\,|\,x') = G_{\mathrm{c}}(t,x\,|\,x') + G_{\mathrm{d}}(t,x\,|\,x'). \tag{16}$$

The following result gives the main properties of these objects.

Proposition 1. *The function $g(x,x',k)$ exists and is a meromorphic function on the upper half-plane $\mathbb{C}_+$ with poles at the zeroes k_j of $a(k)$ and the residues*

$$\operatorname{Res} g(x,x',k)_{k=k_j} = g_j(x,x') \equiv C_j\psi_j(x)\psi_j(x'), \qquad C_j \equiv \frac{\beta_j}{a'(k_j)}.$$

As $|k| \to \infty$, $g(x,x',k)$ has the asymptotic expansion

$$g(x,x',k) = e^{-ik(x-x')}\tilde{g}(x,x',k), \qquad \tilde{g}(x,x',k) \equiv 1 + \sum_{n=1}^{\infty} \frac{m_n(x,x')}{k^n}, \tag{17}$$

where the coefficients are uniformly bounded.

We are now prepared for the fundamental result.

Theorem 1. *The function $G(t,x\,|\,x')$ is a Green's function for the heat operator L with the potential $V(x) = k_I^2 - u(x)$:*

$$LG \equiv \big(-\partial_t + \partial_{xx} - k_I^2 + u(x)\big)G = -\delta(t)\delta(x-x'). \tag{18}$$

Proof. By direct derivation, we find that

$$LG_{\mathrm{c}} = \frac{\theta(t)}{2\pi} \lim_{L\to\infty} \int_{-L}^{L} e^{-t(l^2+2ik_I l)} \big[\partial_{xx} + (l+ik_I)^2 + u(x)\big] g(x,x',l+ik_I)\,dl - \\ - \frac{\delta(t)}{2\pi} \int_{\mathbb{R}} e^{-t(l^2+2ik_I l)} g(x,x',l+ik_I)\,dl.$$

In view of (10), the first term vanishes identically. Therefore,

$$LG_{\mathrm{c}} = -\frac{\delta(t)}{2\pi} \lim_{L\to\infty} \int_{-L}^{L} g(x,x',l+ik_I)\,dl = -\delta(t)\bigg(\delta(x-x') + i \sum_{\kappa_j \geq |k_I|} g_j(x,x')\bigg).$$

This last equality is a deep result, which we do not prove here, regarding the completeness of the eigenfunctions of the Schrödinger operator. moreover, we trivially have

$$LG_{\mathrm{d}} = i\delta(t) \sum_{\kappa_j \geq |k_I|} g_j(x,x').$$

The analyticity properties of $g(x,x',l)$ can be used to derive another interesting, more useful representation of the Green's function. We obtain the following result.

Result 1. *The Green's function for the operator L in* (18) *can also be written as*

$$G(t,x\,|\,x') = i \sum_{\kappa_j \geq k_I} e^{(\kappa_j^2 - k_I^2)t} g_j(x,x')\theta(-t) + \\ + \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t(l^2+k_I^2)} g(x,x',l)\,dl - i \sum_{k_I > \kappa_j} e^{(\kappa_j^2-k_I^2)t} g_j(x,x')\right]\theta(t). \tag{19}$$

Proof. We consider Cauchy integral (13). The corresponding integral in the Green's function can be transformed such that the integration is over the real axis. For this, we consider a rectangular integration contour Γ_L taken in the clockwise sense with vertices at the points $v_1, v_2, v_3, v_4 \in \mathbb{C}_+$ on the complex upper half-plane where

$$v_1 = -L, \qquad v_2 = L, \qquad v_3 = L + ik_I, \qquad v_4 = -L + ik_I,$$

The contribution of the integrals over the vertical sides is proportional to

$$\frac{1}{2\pi}\int_0^{k_I} e^{-t((L+is)^2+k_I^2)} e^{-i(L+is)(x-x')}\,ds,$$

which tends to zero as $L \to \infty$. We recall that $g(k)$ is meromorphic on $\mathbb{C}_+$. Hence, Cauchy's theorem gives

$$\lim_{L\to\infty}\int_{-L}^{L} e^{-t(l^2+k_I^2)} g(x,x',l)\,dl - \lim_{L\to\infty}\int_{-L}^{L} e^{-t(l^2+2ik_I l)} g(x,x',l+ik_I)\,dl = \\ = \lim_{L\to\infty}\int_{\Gamma_L} e^{-t(z^2+k_I^2)} g(x,x',z)\,dz = 2\pi i \sum_{k_I > \kappa_j} e^{(\kappa_j^2-k_I^2)t} g_j(x,x').$$

This amounts to the claim.

We note that the Green's function has the interesting property that it vanishes exponentially fast as either $|t|$ or $|x|$ tends to ∞; in particular, $G_{\mathrm{c}}(t,x\,|\,x')$ has an asymptotic expansion with the leading term given by

$$G_{\mathrm{c}}(t,x\,|\,x') \approx e^{-(k_I^2+l_0^2)t}\frac{g(x,x',-il_0)}{\sqrt{4\pi t}}\theta(t) + i\bigg(\sum_{\kappa_j \leq |l_0|} - \sum_{\kappa_j < k_I}\bigg) g_j(x,x') e^{(\kappa_j^2-k_I^2)t}\theta(t) \tag{20}$$

as $|t| \to \infty$ with $(x-x')/(2t) \equiv l_0$ fixed.

3. The Green's function for reflectionless potentials

We next consider the case of reflectionless potentials characterized by $\rho(k) = 0$. The simplest of such potentials is the one-bound-state Bargmann potential (or soliton potential) given by

$$u(x) = \frac{2\kappa^2}{\cosh^2 \kappa(x - x_0)},$$

where κ and x_0 are constants. It is well known that the spectral data for this potential consist of just one zero (eigenvalue) $k_1 = i\kappa$ and the norming constant $C_1 \equiv 2i\kappa e^{2\kappa x_0}$. The eigenfunction of the discrete spectrum is

$$\psi_1(x) = \psi_+(x, k_1) = \frac{e^{-\kappa x_0}}{\cosh \kappa(x - x_0)}.$$

The wave functions are

$$\psi_+(x, -k) = \psi_-(x, k) = \frac{\phi_+(x, k)}{a(k)} = e^{-ikx}\left(1 + \frac{C_1 e^{-\kappa x}}{k - i\kappa}\psi_1(x)\right).$$

We hence have

$$g(x, x', k) = e^{ik(x'-x)}\left(1 + g_1(x, x')\left(\frac{e^{\kappa(x'-x)}}{k - i\kappa} - \frac{e^{-\kappa(x'-x)}}{k + i\kappa}\right)\right), \tag{21}$$

and the Green's function involves the evaluation of integral (19). We note that

$$\int_{-\infty}^{\infty} \frac{e^{-tl^2 + il(x'-x))}}{l - i\kappa}\, dl = 2\pi i e^{\kappa^2 t - \kappa(x'-x)} \Phi\left(\frac{x' - x}{\kappa\sqrt{2t}} - \kappa\sqrt{2t}\right), \tag{22}$$

where we define

$$\Phi(x) \equiv \int_{-\infty}^{x} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}.$$

We find that

$$\begin{aligned} G(t, x \,|\, x') = {} & \frac{e^{-k_I^2 t - (x-x')^2/(4t)}}{\sqrt{4\pi t}}\theta(t) + \frac{2\kappa e^{(\kappa^2 - k_I^2)t}}{\cosh \kappa(x - x_0) \cosh \kappa(x' - x_0)} \times \\ & \times \left[\left(\Phi\left(\frac{x' - x}{\sqrt{2t}} + \kappa\sqrt{2t}\right) - \Phi\left(\frac{x' - x}{\sqrt{t}} - \kappa\sqrt{2t}\right)\right)\theta(t) + 1_{\{k_I \le \kappa\}}\right], \end{aligned} \tag{23}$$

where we introduce

$$1_{\{k_I \le \kappa\}} = \begin{cases} 1, & k_I \le \kappa, \\ 0, & k_I > \kappa. \end{cases}$$

It turns out that this construction generalizes to the case of reflectionless potentials. We suppose that $u(x)$ is an N-soliton potential, i.e., that $\rho(k) = 0$ and $a(k)$ has N zeroes with norming constants C_j, $j = 1, \ldots, N$. Then

$$\begin{aligned} G(t, x \,|\, x') = {} & \frac{e^{-k_I^2 t - (x-x')^2/(4t)}}{\sqrt{4\pi t}}\theta(t) + i\sum_j g_j(x, x') e^{(\kappa_j^2 - k_I^2)t} \times \\ & \times \left[\left(\Phi\left(\frac{x' - x}{\sqrt{2t}} - \kappa_j\sqrt{2t}\right) - \Phi\left(\frac{x' - x}{\sqrt{2t}} + \kappa_j\sqrt{2t}\right)\right)\theta(t) + 1_{\{k_I \le \kappa_j\}}\right]. \end{aligned} \tag{24}$$

If $x^2 + x'^2 + t^2 \to \infty$ with $(x' - x)/t \to l_0$, then

$$\frac{x' - x}{\sqrt{2t}} - \kappa\sqrt{2t} \to \pm\infty \quad \text{if } \pm(l_0 - \kappa) > 0,$$

and hence

$$\Phi\left(\frac{x' - x}{\sqrt{2t}} - \kappa\sqrt{2t}\right) - \Phi\left(\frac{x' - x}{\sqrt{2t}} + \kappa\sqrt{2t}\right) \underset{x^2+x'^2+t^2\to\infty}{\longrightarrow} -1_{\{-\kappa<l_0<\kappa\}},$$

where

$$-1_{\{-\kappa<l_0<\kappa\}} \equiv -\theta(\kappa - l_0)\theta(l_0 + \kappa) = 1_{\{\kappa\le|l_0|\}} - 1 = 1_{\{\kappa\le|l_0|\}} - 1_{\{\kappa<k_I\}} - 1_{\{\kappa\ge k_I\}},$$

in exact agreement with formula (20).

The density $f(t, x \,|\, x')$ of the KBM with the killing rate $V(x) = b^2 - u(x) \ge 0$ is recovered from the above ideas. We recall that $f(t, x \,|\, x')$ is interpreted as the density of the position of B_t with the killing time greater than t. Further, it solves (18) with the identification $b = k_I$. We determine it in the case of the one-soliton potential

$$V(x) = b^2 - u(x), \qquad u(x) = \frac{2\kappa^2}{\cosh^2 \kappa x}, \tag{25}$$

where b and κ are constants and $b^2 \ge 2\kappa^2$ (by translational invariance, a further constant could be added). We have the following result.

Result 2. *Let a Brownian motion start at x', with the killing rate given by* (25)*. Then the probability that it has not yet been killed at the time $t > 0$ and is located in the interval $[x, x + dx)$ is $P\big(B_t \in [x, x + dx), \tau > t\big) = f(t, x \,|\, x')\, dx$, where*

$$f(t, x \,|\, x') = \frac{e^{-b^2 t - (x-x')^2/(4t)}}{\sqrt{4\pi t}} + \frac{2\kappa e^{(\kappa^2 - b^2)t}}{\cosh \kappa x \cosh \kappa x'}\left(\Phi\left(\frac{x' - x}{\sqrt{2t}} - \kappa\sqrt{2t}\right) - \Phi\left(\frac{x' - x}{\sqrt{2t}} + \kappa\sqrt{2t}\right)\right). \tag{26}$$

The distribution of the death time is given by

$$P(\tau \le t) = 1 - e^{-b^2 t} + \frac{2\kappa e^{(\kappa^2 - b^2)t}}{\cosh \kappa x'} \int \left(\Phi\left(\frac{x' - x}{\sqrt{2t}} + \kappa\sqrt{2t}\right) - \Phi\left(\frac{x' - x}{\sqrt{2t}} - \kappa\sqrt{2t}\right)\right) \frac{dx}{\cosh \kappa x}.$$

The probability that B_t is eventually killed is

$$P(\tau < \infty) = \lim_{t\to\infty} P(\tau \le t) = 1.$$

Proof. The result follows from (24). We note that the requirement $V(x) \ge 0$ yields the constraint on the parameters $b^2 \ge 2\kappa^2$ and several terms in (24) hence drop out yielding a causal Green's function $f(t, x \,|\, x')$.

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