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Squeezing and Quantum Canonical Transformations

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In this paper we present an approach to quantum mechanical canonical transformations. Our main result is that time-dependent quantum canonical transformations can always be cast in the form of squeezing operators. We revise the main properties of these operators in regard to its Lie group properties, how two of them can be combined to yield another operator of the same class and how can also be decomposed and fragmented. In the second part of the paper we show how this procedure works extremely well for the time-dependent quantum harmonic oscillator. The issue of the systematic construction of quantum canonical transformations is also discussed along the lines of Dirac, Wigner, and Schwinger ideas and to the more recent work by Lee. The main conclusion is that the classical phase space transformation can be maintained in the operator formalism but the construction of the quantum canonical transformation is not clearly related to the classical generating function of a classical canonical transformation. We hereby propose the much more efficient method given by the squeezing operators. This method has also been proved to be very useful, by one of the authors, in the framework of the dynamical symmetries (Cerveró, J. M. (1999). *International Journal of Theoretical Physics* **38**, 2095–2109).

KEY WORDS: Quantum mechanics; squeezing; quantum canonical transformations.

1. INTRODUCTION

The purpose of this paper can be simply stated: A time-dependent unitary operator $W(t)$ transforming a given hamiltonian $H(t)$ in another hamiltonian $\tilde{H}(t)$ in the form

$$W(t)H(t)W^{\dagger}(t) - i\hbar W(t)\dot{W}^{\dagger}(t) = \tilde{H}(t)$$

can always be written as a combination of squeezing operators. The operation above described is usually associated to a time-dependent canonical transformation (TDCT). If we succeed in convincing the skeptical reader that a TDCT can always be realized in the quantum formalism by means of a squeezing transformation (ST)

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so that, reciprocally

$$\text{TDCT} \Leftrightarrow \text{ST}$$

Then the rest of the paper will merely be algebra and rhetoric.

The subject of quantum canonical transformations was initiated by Dirac (1930, 1945) and subsequently developed by Wigner (1932) and Schwinger (1951, 1953). In all these cases, however, the formalism was mainly constrained to the time-independent case. Much more recently, the work of Lee (1995), Lee and l'Yi (1995), and Kim and Lee (1999) has added quite a lot of steam to the subject but mainly with an eye on laying the foundations of an unambiguous quantum Hamilton–Jacobi formalism. In the same spirit one should also consider the work of Lewis *et al*. (1996), whose goal seems to be addressed to the construction of the quantum analog to the classical action–angle formalism. Although our aim has a different motivation it turns out to share much more than we previously thought with the above-mentioned approaches.

This paper is divided into two parts. First we shall review the properties of the squeezing operators keeping in mind that excellent reviews are already available in the literature. This is why we shall emphasize mainly only two groups of properties of these operators: fragmentation and multiplication, or in a more colloquial way, breaking them and gluing them back together.

In the second part we shall be discussing the nature of the $W(t)$ operators. The aim is to show that every $W(t)$ operator can be constructed by a simple rule of multiplication of squeezing operators. At this point we shall need a great deal of the properties obtained in the previous section. The main example for illustrating the procedure will be the time-dependent harmonic oscillator, and many of the manipulations will be carried out for the benefit of the reader using this physical system. As it has been mentioned previously one may need to use some already existing reviews and previously known results. For the squeezing operators, we recommend the review by Teich and Saleh (1989) among many other existing excellent papers with similar tutorial approach and contents. The time-dependent harmonic oscillator has been treated throughout in Cerveró and Lejarreta (1990) and many of the properties we shall be using can be found either in Cerveró and Lejarreta (1990) or in the more recent account of Cerveró (1999), which can be considered as the first part of this paper.

2. SQUEEZING OPERATORS

Let the SU(1, 1) Lie algebra be defined through its commutation relations:

$$[K_+, K_-] = -2K_0 \qquad [K_0, K_\pm] = \pm K_\pm \tag{1}$$

A squeezing operator shall be defined henceforth in the following general way:

$$S(\theta, \rho) = \exp\{2i\theta(t)K_0\} \exp\{\rho(t)K_+ - \rho^*(t)K_-\} \tag{2}$$

One should emphasize at this point that this construction can also be generalized to SO(3) or any other Lie algebra (Cerveró and Lejarreta, 1996). However, we shall be considering just the SU(1,1) case for reasons that will become clear just below. Actually the most popular and practical realization of the SU(1, 1) Lie algebra is the one given by the creation and annihilation operators of the canonical algebra by means of the obvious identification

$$K_+ = \frac{1}{2}a^{+^2} \quad K_- = \frac{1}{2}a^2 \quad K_0 = \frac{1}{2}\left(a^+a + \frac{1}{2}\right) \tag{3}$$

Aside from this particular realization of the SU(1,1) Lie algebra, we list some of the main properties of the squeezing operators as defined in (2). First we note that the following transposition property holds:

$$\exp\{2i\theta K_0\}\exp\{\rho K_+ - \rho^* K_-\} = \exp\{\rho\, e^{2i\theta} K_+ - \rho^*\, e^{-2i\theta} K_-\}\ \exp\{2i\theta K_0\} \tag{4}$$

A very important piece of information is the way in which these operators can be multiplied giving rise to another operator of the same kind. This is an obvious consequence of the group law. The first crucial relationship can be written as

$$\exp\{\rho_2 K_+ - \rho_2^* K_-\}\exp\{\rho_1 K_+ - \rho_1^* K_-\} = \exp\{2i\theta_0 K_0\}\exp\{\rho_0 K_+ - \rho_0^* K_-\} \tag{5}$$

After establishing the one to one correspondence

$$\rho_1 = r_1 \exp\{i\varphi_1\} \Rightarrow \eta_1 = \tanh\{r_1\}\exp\{i\varphi_1\} \tag{6}$$

$$\rho_2 = r_2 \exp\{i\varphi_2\} \Rightarrow \eta_2 = \tanh\{r_2\}\exp\{i\varphi_2\} \tag{7}$$

one can obtain θ_0 and ρ_0 in the form

$$\eta_0 = \frac{\eta_1 + \eta_2}{1 + \eta_1^* \eta_2} = |\eta_0| \exp\{i \arg \eta_0\} \tag{8}$$

$$\rho_0 = \operatorname{argtanh}\{|\rho_0|\}\exp\{i \arg \eta_0\} \tag{9}$$

$$2i\theta_0 = \log\left\{\frac{1 + \eta_1^* \eta_2}{1 + \eta_1 \eta_2^*}\right\} \tag{10}$$

Another useful group of properties are those having to do with "fragmentation." Let us take the squeezing operator given by (2). One can easily show that it can also be written as (Gerry and Silverman, 1982)

$$\begin{aligned} S(\theta(t), \rho(t)) &= \exp\{2i\theta(t)K_0\}\exp\{\rho(t)K_+ - \rho^*(t)K_-\} \\ &= \exp\{2i\theta(t)K_0\}\exp\{\eta(t)K_+\}\exp\{\gamma(t)K_0\}\exp\{-\eta(t)K_-\} \end{aligned} \tag{11}$$

where $\eta(t)$ and $\gamma(t)$ are given by correspondences of similar sort of those listed in (6) and (7), namely

$$\rho(t) = r(t)\exp\{i\varphi(t)\} \Rightarrow \eta(t) = \tanh\{r(t)\}\exp\{i\varphi(t)\} \tag{12}$$

$$\gamma(t) = \ln(1 - |\eta(t)|^2) \tag{13}$$

Finally the transformation of the Lie algebra generators under the squeezing operators are also extremely useful for some of the calculations presented below

$$S(\theta, \rho)K_0 S^\dagger(\theta, \rho) = \cosh\{2r(t)\}K_0 - \frac{1}{2}\sinh\{2r(t)\} \times \left\{e^{i(\theta_+ + \theta_-)}K_+ + e^{-i(\theta_+ + \theta_-)}K_-\right\} \tag{14}$$

$$S(\theta, \rho)K_+ S^\dagger(\theta, \rho) = \cosh^2\{r(t)\}\, e^{2i\theta_+} K_+ + \sinh^2\{r(t)\}\, e^{-2i\theta_-} K_- - e^{i(\theta_+ - \theta_-)}\sinh\{2r(t)\}K_0 \tag{15}$$

$$S(\theta, \rho)K_- S^\dagger(\theta, \rho) = \sinh^2\{r(t)\}\, e^{2i\theta_-} K_+ + \cosh^2\{r(t)\}\, e^{-2i\theta_+} K_- - e^{i(\theta_- - \theta_+)}\sinh\{2r(t)\}K_0 \tag{16}$$

where

$$\theta_+ = \theta(t) \quad \text{and} \quad \theta_- = \varphi(t) + \theta(t) \tag{17}$$

3. TIME-DEPENDENT CANONICAL TRANSFORMATIONS

Suppose we start with a time-dependent hamiltonian and we apply a sequence of time-dependent canonical transformations in such a way that we go from the initial $H(t)$ to the final $H_0(t)$ through the following set of TDCT:

$$H(t) \Rightarrow H_1(t) \Rightarrow H_{12}(t) = H_0(t)$$

This sequence really means in terms of the actual application of the $W(t)$ operators the following set of step-by-step transformations:

$$W_1(t)H(t)W_1^\dagger(t) - i\hbar W_1(t)\dot{W}_1^\dagger(t) = H_1(t) \tag{18}$$

$$W_2(t)H_1(t)W_2^\dagger(t) - i\hbar W_2(t)\dot{W}_2^\dagger(t) = H_{12}(t) = H_0(t) \tag{19}$$

There must be an operator that does the same job in just one step, namely

$$W(t)H(t)W^\dagger(t) - i\hbar W(t)\dot{W}^\dagger(t) = H_0(t) \tag{20}$$

This operator must necessarily be $W(t) = W_2(t)W_1(t)$, as one can easily check by merely using the time-dependent Schrödinger equation. This obvious fact suggests a group law. Let this group be SU(1, 1). As a general example for quadratic time-dependent hamiltonians we choose $H(t)$ to be

$$H(t) = \frac{1}{2m}\left\{\beta_3(t)\hat{p}^2 + \beta_2(t)m\omega_0[\hat{x}\hat{p} + \hat{p}\hat{x}] + \beta_1(t)m^2\omega_0^2\hat{x}^2\right\} \tag{21}$$

And we choose as $W_1(t)$ and $W_2(t)$ the following unitary operators (Cerveró and Lejarreta, 1990):

$$W_1(t) = \exp\left\{\frac{i}{4\hbar}(\ln\beta_3(t))[\hat{x}\hat{p} + \hat{p}\hat{x}]\right\} \tag{22}$$

$$W_2(t) = \exp\left\{\frac{im}{2\hbar}\left(\omega_0\beta_2(t) - \frac{\dot{\beta}_3(t)}{2\beta_3(t)}\right)\hat{x}^2\right\} \tag{23}$$

The resulting hamiltonians in each of the steps (18) and (19) turn out to be

$$H_1(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}\left\{\omega_0\beta_2(t) - \frac{\dot{\beta}_3(t)}{2\beta_3(t)}\right\}[\hat{x}\hat{p} + \hat{p}\hat{x}] + \frac{1}{2}m\omega_0^2\beta_1(t)\beta_3(t)\hat{x}^2 \tag{24}$$

$$\begin{aligned} H_{12}(t) = H_0(t) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\left\{\omega_0^2\left(\beta_1\beta_3 - \beta_2^2\right) + \omega_0\left(\frac{\dot{\beta}_3\beta_2 - \dot{\beta}_2\beta_3}{\beta_3}\right)\right. \\ \left. + \frac{\ddot{\beta}_3}{2\beta_3} - \frac{3\dot{\beta}_3^2}{4\beta_3^2}\right\}\hat{x}^2 \end{aligned} \tag{25}$$

The next step is to show that $W_1(t)$ and $W_2(t)$ can be written in a squeezing operator form. It is not hard to see that $W_1(t)$ can be written as

$$W_1(t) = \exp\{\rho_1(t)K_+ - \rho_1^*(t)K_-\} \tag{26}$$

where we have used the well-known form of the creation and annihilation operators as linear combinations of the canonical operators $\hat{x}$ and $\hat{p}$. The function $\rho_1(t)$ has no complex phase and turns out to be in this particular case just a real function of the form

$$\rho_1(t) = \operatorname{argtanh}\left\{\frac{1 - \beta_3(t)}{1 + \beta_3(t)}\right\} \tag{27}$$

Furthermore, the operator $W_2(t)$ can be written as

$$\begin{aligned} W_2(t) &= \exp\{i\gamma(t)[K_+ + K_- + 2K_0]\} \\ &= \exp\{2i\theta_2(t)K_0\}\exp\{\rho_2(t)K_+ - \rho_2^*(t)K_-\} \end{aligned} \tag{28}$$

where

$$\gamma(t) = \frac{1}{2\omega_0}\left\{\omega_0\beta_2(t) - \frac{\dot{\beta}_3(t)}{2\beta_3(t)}\right\} \tag{29}$$

$$\theta_2(t) = \arctan\gamma(t) \tag{30}$$

$$\rho_2(t) = \operatorname{argtanh}\left\{\frac{\gamma(t)}{(1+\gamma^2(t))^{1/2}}\right\}\exp\left\{i\left(\frac{\pi}{2}-\theta_2(t)\right)\right\} \tag{31}$$

Obviously (26) and (28) are squeezing operators as the one generally defined in (2) and can be written in this notation as $S(0, \rho_1(t))$ and $S(\theta_2(t), \rho_2(t))$. It is now just a matter of a tedious but trivial calculation to combine them by making use of the expressions given in the previous section. What we try to do is basically to use the expressions (5)–(10) in order to obtain a single squeezing operator in the form

$$\begin{aligned} W(t) &= W_2(t)W_1(t) \\ &= \exp\{2i\theta_2(t)K_0\}\exp\{\rho_2(t)K_+ - \rho_2^*(t)K_-\}\exp\{\rho_1(t)K_+ - \rho_1^*(t)K_-\} \\ &= \exp\{2i\theta_0 K_0\}\exp\{\rho_0(t)K_+ - \rho_0^*(t)K_-\} \end{aligned} \tag{32}$$

where in this case

$$\begin{aligned} \rho_0(t) = \operatorname{argtanh}&\left\{\sqrt{\frac{4\gamma^2(t)+(1-\beta_3(t))^2}{4\gamma^2(t)+(1+\beta_3(t))^2}}\right\}\exp\left\{i\left(\arctan\left\{\frac{2\gamma(t)}{1-\beta_3(t)}\right\}\right.\right. \\ &\left.\left. -\arctan\left\{\frac{2\gamma(t)}{1+\beta_3(t)}\right\}\right)\right\} \end{aligned} \tag{33}$$

$$\theta_0(t) = \arctan\left\{\frac{2\gamma(t)}{1+\beta_3(t)}\right\} \tag{34}$$

In the final part of this section we shall be discussing the relationship of what so far has been done to relate squeezing operators and time-dependent quantum canonical transformations with what is believed to be the standard lore on the connection between quantum and classical canonical transformations. As it has been previously mentioned the beginning of this sort of discussions is due to Dirac (1930, 1945), Wigner (1932), and Schwinger (1951, 1953). Recently a renewed interest on the subject has been put forward by Lee (1995), Lee and l'Yi (1995), and Kim and Lee (1999). Also we should mention that quite recently Kim and Wigner (1990) mentioned the relationship between squeezing and canonical transformations in the context of the time-independent quantum problem, cleverly relating the squeezing properties to 2+1-Lorentz transformations because of the well-known fact that SO(2, 1) (the Lorentz group in "flatland") is locally isomorphic to SU(1, 1). In the rest of this paper we shall be dealing with the classical time-dependent canonical transformations of the Hamiltonian

$$H(t) = \frac{1}{2m}\left\{\beta_3(t)p^2 + \beta_2(t)m\omega_0[xp+px] + \beta_1(t)m^2\omega_0^2x^2\right\} \tag{35}$$

which is the classical counterpart of (2). It is a trivial exercise in classical mechanics (Goldstein, 1959) to find a generating function of a canonical transformation in phase space that leads (35) to

$$H_0(t) = \frac{1}{2m}\{P^2 + m^2\Omega^2(t)X^2\} \tag{36}$$

where $\Omega^2(t)$ is given by the expression

$$\Omega^2(t) = \omega_0^2\{\beta_1\beta_3 - \beta_2^2\} + \omega_0\left\{\frac{\dot{\beta}_3\beta_2 - \dot{\beta}_2\beta_3}{\beta_3}\right\} + \left\{\frac{1}{2}\left(\frac{\ddot{\beta}_3}{\beta_3} - \frac{\dot{\beta}_3^2}{\beta_3^2}\right) - \frac{1}{4}\left(\frac{\dot{\beta}_3^2}{\beta_3^2}\right)\right\} \tag{37}$$

which obviously coincides with the one appearing in (25). In classical mechanics a generation function of Class 2, $F_2^{\mathrm{C}}(x, P, t)$, can be constructed such that

$$\frac{\partial F_2^{\mathrm{C}}(x, P, t)}{\partial x} = p; \quad \frac{\partial F_2^{\mathrm{C}}(x, P, t)}{\partial P} = X; \quad H_0(t) = H(t) + \frac{\partial F_2^{\mathrm{C}}(x, P, t)}{\partial t} \tag{38}$$

The explicit form of $F_2^{\mathrm{C}}(x, P, t)$ in our case, takes the form

$$F_2^{\mathrm{C}}(x, P, t) = \frac{m}{2\beta_3(t)}\left\{\frac{\dot{\beta}_3(t)}{2\beta_3(t)} - \omega_0\beta_2(t)\right\}x^2 + \beta_3(t)^{-\frac{1}{2}}xP \tag{39}$$

giving rise to the following canonical phase space transformation:

$$X = \beta_3(t)^{-\frac{1}{2}}x \tag{40}$$

$$P = \beta_3(t)^{\frac{1}{2}}\left\{p + \left(\frac{m}{\beta_3(t)}\right)\left(\omega_0\beta_2(t) - \frac{\dot{\beta}_3(t)}{2\beta_3(t)}\right)x\right\} \tag{41}$$

The main question arises as to whether these classical phase space transformations that are under the basis of a canonical transformation have anything to do with the squeezing formalism. The idea of constructing the quantum canonical operator by merely using the generating function as Dirac (1930, 1945) suggested is simply too naive and does not work: we cannot reproduce $W(t)$ given by (32) (together with (33) and (34)) just by constructing the dimensionless exponential of the classical generating function (39):

$$\exp\left\{\frac{i}{\hbar}F_2^{\mathrm{C}}(x, P, t)\right\} \neq W(t) \tag{42}$$

With this result in mind one is tempted to conjecture that a quantum generating function $F_2^{\mathrm{Q}}(x, P, t)$ could be defined in the same spirit as the Feynman quantum action used to construct the quantum propagator. Such $F_2^{\mathrm{Q}}(x, P, t)$ should retain some features of the $F_2^{\mathrm{C}}(x, P, t)$, in particular the classical limit that has to be properly defined. It is also encouraging that although these generating functions obviously differ, our time-dependent quantum canonical operator $W(t)$ yields the phase space transformation (40)–(41), which is clearly classical in origin but can be written in the language of canonical operators. In fact one can actually check that the following relationships hold:

$$W^{\dagger}(t)\hat{x}W(t) = \hat{X} = \beta_3(t)^{-\frac{1}{2}}\hat{x} \tag{43}$$

$$W^{\dagger}(t)\hat{p}W(t) = \hat{P} = \beta_3(t)^{\frac{1}{2}}\left\{\hat{p} + \left(\frac{m}{\beta_3(t)}\right)\left(\omega_0\beta_2(t) - \frac{\dot{\beta}_3(t)}{2\beta_3(t)}\right)\hat{x}\right\} \tag{44}$$

The conditions expressed by the above equations have been already stressed by Lee and l'Yi (1995) and Kim and Lee (1999) as the actual main requirements for a quantum canonical transformation rather than the dubious association (42). We see in this time-dependent analysis that the classical phase space transformation still survives in the quantum operator domain. To go further we conjecture that the squeezing operator formalism will be much more fruitful in view of the results hereby presented and an extension of these will be presented in a future report now in preparation.

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