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Coalescence limits for higher order Painlevé equations

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Abstract

It is well-known that the first Painlevé equation arises as a coalescence limit of each of the other five Painlevé equations. This result is important because it shows that, since the solution of the first Painlevé equation cannot be expressed in terms of known functions, then neither can the solutions of the other five Painlevé equations (except possibly for special values of their parameters). Here we derive analogous results for three recently derived higher order ordinary differential equations believed to define new transcendental functions. We show that each of the equations considered has as a coalescence limit a member of the first Painlevé hierarchy. We thus reduce the problem of showing that the solutions of these three cannot be expressed in terms of known functions to that of showing that the same is true for the corresponding first Painlevé equations. This represents the first extension of coalescence results for the Painlevé equations to their higher order analogues.

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1. Introduction

The search for new functions defined as solutions of differential equations led, at the turn of the last century, to the discovery of the six Painlevé equations [1–4]. One question of particular importance, and indeed of some controversy, was that of whether the solutions of these equations could be expressed in terms of known functions. Since it could be shown that the first Painlevé equation arises as a coalescence limit of the other five (see [4]), this question was reduced to that of showing that the solution of the first Painlevé equation defines a new transcendent. This last problem has in fact only been solved remarkably recently [5–7].

As an example of a coalescence limit, let us consider that between the second Painlevé equation P_{II}, in $v(y)$ with arbitrary parameter a,

$$v_{yy} - 2v^3 - yv - a = 0, \tag{1}$$

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and the first Painlevé equation P_{I}, in $u(x)$,

$$u_{xx} + 3u^2 + x = 0. \tag{2}$$

Making the change of variables (a rescaled version of that in [4])

$$v(y) = \varepsilon u(x) - \frac{1}{8\varepsilon^5}, \qquad y = 2\varepsilon^2 x - \frac{3}{32\varepsilon^{10}}, \qquad a = -\frac{1}{128\varepsilon^{15}}, \tag{3}$$

in (1) leads to the equation

$$u_{xx} + 3u^2 + x - 8\varepsilon^6\big(u^2 + x\big)u = 0, \tag{4}$$

which in the limit $\varepsilon \to 0$ gives (2). Thus we see that P_{II} contains P_{I} as a coalescence limit. Coalescence limits of the six Painlevé equations may be summarized as [4,8]

$$\begin{array}{ccccc} & & \longrightarrow P_{\mathrm{IV}} \longrightarrow & & \\ P_{\mathrm{VI}} \longrightarrow P_{\mathrm{V}} & & & & P_{\mathrm{II}} \longrightarrow P_{\mathrm{I}} \\ & & \longrightarrow P_{\mathrm{III}} \longrightarrow & & \end{array} \tag{5}$$

and thus we see that P_{I} can be obtained from each of the other Painlevé equations.

It is this limiting process that we seek to explore here for higher order analogues of the Painlevé equations. In particular, we will seek linear transformations of dependent and independent variables, and also of parameters a_i (into new parameters $A_1, A_2, \ldots, A_N$), with coefficients dependent on a parameter ε,

$$v(y) = \alpha(\varepsilon)u(x) + \beta(\varepsilon), \qquad y = \delta(\varepsilon)x + \gamma(\varepsilon), \qquad a_i = \sum_{j=1}^{N} \lambda_j(\varepsilon)A_j + \mu(\varepsilon). \tag{6}$$

We note that all the transformations used in the coalescence processes in (5) are of this form [4] (see, e.g., (3)). It turns out that consideration of such linear transformations is sufficient for the purposes of the present Letter. Our requirement is that the expression obtained by solving the transformed equation for the highest derivative of u be analytic in ε at $\varepsilon = 0$; the limit $\varepsilon \to 0$ then gives our coalescence limit.

There are two reasons why our results are important. First, we extend the analogy between certain Painlevé equations and their higher order analogues. Second, as with Painlevé's results [4], we reduce the number of equations for which it must be shown that their solution cannot be expressed in terms of known functions. This last is of great practical importance in the study of higher order Painlevé equations.

Higher order Painlevé equations may be obtained in a variety of ways. One is by taking similarity reductions of the higher order members of a hierarchy of completely integrable partial differential equations; thus, for example, the modified Korteweg–de Vries hierarchy yields the P_{II} hierarchy [9]. Another is by extending the classification programme of Painlevé to higher order differential equations [10]. A third approach is that developed in [11–14]. Here we consider coalescence limits for certain higher order analogues of the second Painlevé equation.

2. Examples from the generalized second Painlevé hierarchy

Here we consider two examples from a generalized version of the P_{II} hierarchy,

$$\partial_y^{-1}\bar{R}^n v_y + \partial_y^{-1}\sum_{j=1}^{n-1} b_j \bar{R}^j v_y - yv - a = 0. \tag{7}$$

Here a and all b_j are arbitrary constants, $\partial_y = d/dy$ and

$$\bar{R} = \partial_y^2 - 4v^2 - 4v_y\partial_y^{-1}v, \tag{8}$$

is the recursion operator of the modified Korteweg–de Vries hierarchy. The hierarchy of equations (7) consists of linear combinations of the members of the $P_{\rm II}$ hierarchy given in [9]; setting $n=1$ gives $P_{\rm II}$ (1). We note that here we have assumed that the coefficient of the non-autonomous term is non-zero, in which case this coefficient may be rescaled to -1 and the coefficient b_0 (of a term in v) can be set to zero.

Our aim is to show that members of this hierarchy have as coalescence limits corresponding members of the generalized $P_{\rm I}$ hierarchy

$$\partial_x^{-1}R^n u_x + \partial_x^{-1}\sum_{j=0}^{n-2} B_j R^j u_x + x = 0, \tag{9}$$

where here all B_j are arbitrary constants, $\partial_x = d/dx$ and

$$R = \partial_x^2 + 4u + 2u_x\partial_x^{-1}, \tag{10}$$

is the recursion operator of the Kortweg–de Vries hierarchy. The hierarchy (9) corresponds to a special case ($g_{n-1}=0$) of the hierarchy (3.29) in [12]. As noted in [12] we can without loss of generality set $B_{n-1}=0$; here we assume that the coefficient of the non-autonomous term is non-zero, in which case we may rescale this coefficient to 1 and also set any constant of integration to zero.

As our first example we take $n=2$ in (7), which gives (setting $b_1 = c$)

$$v_{yyyy} - 10v^2 v_{yy} - 10vv_y^2 + 6v^5 + c\big(v_{yy} - 2v^3\big) - yv - a = 0. \tag{11}$$

Making the change of dependent and independent variables given in (6), i.e.,

$$v(y) = \alpha u(x) + \beta, \qquad y = \delta x + \gamma, \tag{12}$$

we obtain, setting $a = 6\beta^5 - 2\beta^3 c - \beta\gamma$ to remove the additional constant term,

$$\begin{aligned} &u_{xxxx} - 10\alpha^2\delta^2u^2u_{xx} - 20\alpha\beta\delta^2uu_{xx} + c\delta^2u_{xx} - 10\beta^2\delta^2u_{xx} - 10\alpha^2\delta^2uu_x^2 \\ &\quad - 10\alpha\beta\delta^2u_x^2 + 6\alpha^4\delta^4u^5 + 30\alpha^3\beta\delta^4u^4 - 2\alpha^2\delta^4cu^3 + 60\alpha^2\beta^2\delta^4u^3 - 6\alpha\beta\delta^4cu^2 \\ &\qquad + 60\alpha\beta^3\delta^4u^2 - \delta^5xu - \delta^4\gamma u - 6\beta^2\delta^4cu + 30\beta^4\delta^4u - \big(\beta\delta^5/\alpha\big)x = 0. \end{aligned} \tag{13}$$

In this last we now set $\beta = -1/(2\alpha\delta^2)$ and $c = 5/(2\alpha^2\delta^4)$, which then yields the sought-after dominant terms (the first four terms of the following),

$$\begin{aligned} &u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3 - 10\alpha^2\delta^2u^2u_{xx} - 10\alpha^2\delta^2uu_x^2 + 6\alpha^4\delta^4u^5 \\ &\quad - 15\alpha^2\delta^2u^4 - \delta^5xu - \delta^4\gamma u - \big(15/\big(8\alpha^4\delta^4\big)\big)u + \big(\delta^3/\big(2\alpha^2\big)\big)x = 0. \end{aligned} \tag{14}$$

In this last equation we see that, in order to obtain the sought-after non-autonomous term, we will have to take $\delta^3 = 2\alpha^2$. We may therefore take either both δ and α as positive powers of ε, or both as negative powers of ε. It is only the former case that allows us to obtain an expression for u_{xxxx} analytic at $\varepsilon = 0$. This expression we obtain by first using γ to remove the term in $-(15/(8\alpha^4\delta^4))u$, by taking $\gamma = -(15/(8\alpha^4\delta^8)) - (B/\delta^4)$, and then setting $\delta = 2\varepsilon^2$ and $\alpha = 2\varepsilon^3$; this gives

$$\begin{aligned} &u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3 + Bu + x \\ &\quad - 16\varepsilon^{10}\big(10u^2u_{xx} + 10uu_x^2 + 15u^4 + 2xu\big) + 1536\varepsilon^{20}u^5 = 0. \end{aligned} \tag{15}$$

Taking the limit $\varepsilon \to 0$ then gives as coalescence limit of (11) the equation

$$u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3 + Bu + x = 0, \tag{16}$$

which is (9) for $n = 2$ (and $B_0 = B$). That is, the second member of the generalized P_{II} hierarchy has as coalescence limit the second member of the generalized P_{I} hierarchy. This is analogous to the case $n = 1$ of these two hierarchies, which is precisely the example given in Section 1 ((7) with $n = 1$ is (1), and (9) is (2)).

In summary, the change of variables used to obtain this coalescence limit is

$$v(y) = 2\varepsilon^3 u(x) - \frac{1}{2^4\varepsilon^7}, \qquad y = 2\varepsilon^2 x - \frac{15}{2^{15}\varepsilon^{28}} - \frac{B}{2^4\varepsilon^8},$$
$$c = \frac{5}{2^7\varepsilon^{14}}, \qquad a = -\frac{1}{2^{16}\varepsilon^{35}} - \frac{B}{2^8\varepsilon^{15}}, \tag{17}$$

which then yields (15) and in the limit $\varepsilon \to 0$ (16).

We now make two remarks on the above calculation. The first is that it is only by including the non-dominant terms $c(v_{yy} - 2v^3)$ in (11) that we are able to correct the coefficient of u^3 to be as in (14) above. That is, if we were to take the P_{II} hierarchy to be as in (7) but with all $b_j = 0$, as in [9], then we would not be able to obtain the sought-after coalescence limit. It is for this reason that we have considered the generalized P_{II} hierarchy presented here, i.e., (7), rather than that in [9]. We note that this case $n = 2$ of (7), i.e., (11), can be found in [15]. Our second remark is that the equation we have obtained as a coalescence limit is the second member of our generalized P_{I} hierarchy (9), but that equally we could have obtained as coalescence limit equation (16) with $B = 0$, i.e., the second member of the original P_{I} hierarchy (all $B_j = 0$ in (9)) as given by Kudryashov [16].

As our second example we take the case $n = 3$ of (7), with $b_1 = c$ and $b_2 = d$,

$$v_{yyyyyyy} - 14v^2 v_{yyyyy} - 56vv_y v_{yyy} - 42vv_{yy}^2 - 70v_y^2 v_{yy} + 70v^4 v_{yy} + 140v^3 v_y^2$$
$$- 20v^7 + d\big(v_{yyyyy} - 10v^2 v_{yy} - 10vv_y^2 + 6v^5\big) + c\big(v_{yy} - 2v^3\big) - yv - a = 0. \tag{18}$$

We make as before the change of dependent and independent variables given in (6), i.e., (12), and seek α, β, γ, δ, d, c and a in terms of a parameter ε such that the resulting equation, when solved for $u_{xxxxxxx}$, gives an expression analytic at $\varepsilon = 0$, and has as the limit $\varepsilon \to 0$ Eq. (9) with $n = 3$. We do not give details of our calculation here, but summarize our result as follows. The change of variables

$$v(y) = 4\varepsilon^5 u(x) - \frac{1}{2^5\varepsilon^9}, \qquad y = 2\varepsilon^2 x - \frac{35}{2^{28}\varepsilon^{54}} - \frac{B}{2^6\varepsilon^{12}} - \frac{3C}{2^{13}\varepsilon^{26}},$$
$$d = \frac{7}{2^9\varepsilon^{18}}, \qquad c = \frac{35}{2^{19}\varepsilon^{36}} + \frac{C}{2^4\varepsilon^8}, \qquad a = -\frac{1}{2^{29}\varepsilon^{63}} - \frac{B}{2^{11}\varepsilon^{21}} - \frac{C}{2^{17}\varepsilon^{35}}, \tag{19}$$

made in (18), yields the equation

$$u_{xxxxxx} + 14uu_{xxxx} + 28u_x u_{xxx} + 21u_{xx}^2 + 70u^2 u_{xx} + 70uu_x^2 + 35u^4 + C\big(u_{xx} + 3u^2\big) + Bu + x$$
$$- 128\varepsilon^{14}\big(7u^2 u_{xxxx} + 28uu_x u_{xxx} + 21uu_{xx}^2 + 35u_x^2 u_{xx} + 70u^3 u_{xx} + 105u^2 u_x^2 + 42u^5 + Cu^3 + xu\big)$$
$$+ 286720\varepsilon^{28}\big(u^4 u_{xx} + 2u^3 u_x^2 + u^6\big) - 5242880\varepsilon^{42} u^7 = 0, \tag{20}$$

which, in the limit $\varepsilon \to 0$, gives

$$u_{xxxxxx} + 14uu_{xxxx} + 28u_x u_{xxx} + 21u_{xx}^2 + 70u^2 u_{xx} + 70uu_x^2 + 35u^4$$
$$+ C\big(u_{xx} + 3u^2\big) + Bu + x = 0. \tag{21}$$

This last is (9) for $n = 3$, with $B_1 = C$ and $B_0 = B$.

Similar remarks hold as for our previous example: that we could not have achieved the sought-after coalescence limit if instead of the generalized case (18) we had considered the third member of the P_{II} hierarchy in [9] (i.e., the case $d = c = 0$ here); and that we could have obtained as a coalescence limit the corresponding member of the non-generalized P_{I} hierarchy (by taking $B = C = 0$ in the above).

Clearly we expect that higher order members of the generalized P_{II} hierarchy (7) have as coalescence limits corresponding higher order members of the generalized P_{I} hierarchy (9), i.e., including all non-dominant terms with coefficients $B_0, \ldots, B_{n-2}$.

3. A new higher order second Painlevé equation

In this section we consider the second member of an alternative P_{II} hierarchy presented in [14]. This equation was originally derived as a system of equations,

$$\frac{1}{4}\left(v_{yy} - 3vv_y + v^3 + 6vw\right) + cv + g_3 y - \gamma_2 = 0, \tag{22}$$

$$\frac{1}{4}\left(w_{yy} + 3w^2 + 3vw_y + 3v^2 w\right) + cw - \delta_2 = 0, \tag{23}$$

where g_3, c, γ_2 and δ_2 are all constants. The system (22), (23) has the underlying linear problem

$$\mathbf{\Psi}_y = \mathcal{F}\mathbf{\Psi}, \qquad \frac{1}{2}g_3\mathbf{\Psi}_\lambda = \mathcal{H}\mathbf{\Psi}, \tag{24}$$

where

$$\mathcal{F} = \begin{pmatrix} -\lambda + \frac{1}{2}v & 1 \\ -w & \lambda - \frac{1}{2}v \end{pmatrix}, \tag{25}$$

$$\mathcal{H} = \frac{1}{4}\left(\begin{array}{l|l} -w_y - 2vw - 2\lambda w & 2w - v_y + v^2 \\ \quad - 2g_3 y - 4\lambda^3 - 4c\lambda + 2\gamma_2 & \quad + 2\lambda v + 4\lambda^2 + 4c \\ \hline vw_y - 2\lambda w_y - wv_y & w_y + 2vw + 2\lambda w \\ \quad + w^2 + 2v^2 w - 2\lambda vw & \quad + 2g_3 y + 4\lambda^3 + 4c\lambda - 2\gamma_2 \\ \quad - 4\lambda^2 w - 4\delta_2 & \end{array}\right). \tag{26}$$

Solving (22) for w and substituting in (23) then gives a fourth order equation for v, the linear problem for which given in [14] is then obtained from (25), (26).[1]

It is the case $g_3 \neq 0$ that defines our higher order analogue of the second Painlevé equation, and it is this case that we consider here. Since $g_3 \neq 0$, we may assume without loss of generality that $\gamma_2 = 0$ and $g_3 = 1$. The corresponding fourth order equation for v, where we also set $\delta_2 = (b/24) - (c^2/3) - (1/2)$, is then given by

$$\begin{aligned} v_{yyyy} = {} & 2\frac{v_{yyy}v_y}{v} + \frac{3}{2}\frac{v_{yy}^2}{v} - 2\frac{v_{yy}v_y^2}{v^2} + 5v^2 v_{yy} + 8y\frac{v_{yy}}{v} + \frac{5}{2}vv_y^2 \\ & - 8y\frac{v_y^2}{v^2} + 8\frac{v_y}{v} - \frac{5}{2}v^5 - 12cv^3 - 8yv^2 - bv + 8\frac{y^2}{v}. \end{aligned} \tag{27}$$

We now consider obtaining a coalescence limit of Eq. (27). We make the change of dependent and independent variables given in (6), i.e., (12), and seek α, β, γ, δ, c and b such that the resulting equation, when solved for u_{xxxx}, gives an expression analytic at $\varepsilon = 0$, and has as the limit $\varepsilon \to 0$, as might be expected, the fourth order member of the first Painlevé hierarchy (9), or a special case thereof.

We do not give details of our calculations here, but summarize our result as follows. The change of variables

$$v(y) = 2\varepsilon u(x) - \frac{1}{3\varepsilon^7}, \qquad y = \varepsilon^3 x + \frac{5}{216\varepsilon^{21}}, \qquad c = -\frac{5}{96\varepsilon^{14}}, \qquad b = \frac{5}{36\varepsilon^{28}}, \tag{28}$$

[1] Note there is a missing factor of $1/(6u)$ in the lower left component of $\mathcal{F}$ given in (99) in [14].

made in (27), yields the equation

$$\begin{aligned} &u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3 \\ &\quad + x - \varepsilon^8\big(12uu_{xxxx} - 12u_xu_{xxx} - 9u_{xx}^2 + 120u^2u_{xx} + 30uu_x^2 + 145u^4 + 14xu\big) \\ &\quad + 6\varepsilon^{16}\big(6u^2u_{xxxx} - 12uu_xu_{xxx} - 9uu_{xx}^2 + 12u_x^2u_{xx} + 80u^3u_{xx} + 4xu_{xx} + 30u^2u_x^2 \\ &\qquad\qquad + 4u_x + 125u^5 + 16xu^2\big) \\ &\quad - 12\varepsilon^{24}\big(60u^4u_{xx} + 12xuu_{xx} + 30u^3u_x^2 - 12xu_x^2 + 12uu_x + 140u^6 + 32xu^3 - x^2\big) \\ &\quad + 72\varepsilon^{32}\big(20u^7 + 8xu^4 - x^2u\big) = 0, \end{aligned} \tag{29}$$

which, in the limit $\varepsilon \to 0$, gives

$$u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3 + x = 0. \tag{30}$$

Thus we see that the second member of our alternative P_{II} hierarchy, i.e., Eq. (27), has as a coalescence limit a special case ($B_0 = 0$) of the second member ($n = 2$) of the P_{I} hierarchy (9). That is, Eq. (27) has as a coalescence limit the second member of the original P_{I} hierarchy of Kudryashov [16]. (In fact, (28) is not the only change of variables which yields (30) as a coalescence limit of (27).)

4. Conclusions

In this Letter we have considered coalescence limits for certain members of two distinct P_{II} hierarchies. We have shown that these equations have as coalescence limits corresponding members of the first Painlevé hierarchy. Thus we at once both extend the analogy between the higher order analogues of the second and first Painlevé equations with those equations themselves, and at the same time reduce the number of equations for which it must be shown that their solutions cannot be expressed in terms of known functions. Thus any proof that the higher order first Painlevé equations (generalized or not) define new transcendents now also serves to show that neither can the general solutions of (the general case of) the higher order second Painlevé equations studied here be expressed in terms of known functions.

Other aspects of coalescence limits will be addressed in subsequent papers.

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