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EXTENDED ROTATION AND SCALING GROUPS FOR NONLINEAR EVOLUTION EQUATIONS

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A (1+1)-dimensional nonlinear evolution equation is invariant under the rotation group if it is invariant under the infinitesimal generator $V = x\partial_u - u\partial_x$. Then the solution satisfies the condition $u_x = -x/u$. For equations that do not admit the rotation group, we provide an extension of the rotation group. The corresponding exact solution can be constructed via the invariant set $R_0 = \{u\colon u_x = xF(u)\}$ of a contact first-order differential structure, where F is a smooth function to be determined. The time evolution on R_0 is shown to be governed by a first-order dynamical system. We introduce an extension of the scaling groups characterized by an invariant set $\widetilde{S}_0$ that depends on two constants ϵ and $n \neq 1$. When $\epsilon = 0$, it reduces to the invariant set S_0 introduced by Galaktionov. We also introduce a generalization of both the scaling and rotation groups, which is described by an invariant set E_0 with parameters a and b. When $a = 0$ or $b = 0$, it respectively reduces to R_0 or S_0. These approaches are used to obtain exact solutions and reductions of dynamical systems of nonlinear evolution equations.

Keywords: differential evolution equations, rotation group, scaling group

1. Introduction

We consider an (1+1)-dimensional kth-order nonlinear evolution equation of the form

$$u_t = E(x, u, u_1, \ldots, u_k), \tag{1}$$

where $u_t = \partial u/\partial t$ is the time derivative and u_i denotes the ith partial derivatives $\partial^i u/\partial x^i$ with respect to the spatial variable x. The function E is a smooth function of the indicated variables. The solutions of Eq. (1) are assumed to be smooth.

Several developed methods related to the symmetry group theory have been used to construct exact solutions of nonlinear partial differential equations (PDEs). They include the classical symmetry group method [1], the nonclassical symmetry group method [2], the generalized conditional symmetry approach [3], the direct method [4], the differential constraint method [5], and the sign-invariant and invariant-subspace approach [6], [7]. These methods are related in some sense.

In [8], Galaktionov proposed a nonlinear extension of the ordinary scaling group, which is described by the invariance of the set $S_0 = \{u\colon u_x = F(u)/x\}$. The approach has been used to construct exact solutions of equations of form (1), and it is related to the sign-invariant and invariant-subspace approach as well as the generalized conditional symmetry approach. At the end of [8], he noted that it is important to give a nonlinear extension for other nonscaling groups.

The aim in this paper is to introduce some new extensions of rotation groups and scaling groups. In Sec. 2, we propose a nonlinear extension of the ordinary rotation group in $\mathbb{R}^2$, which is described by the

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invariant set $R_0 = \{u(x) : u_x = xF(u)\}$. In Sec. 3, we introduce a new nonlinear extension of the scaling group, which is described by the invariant set

$$\widetilde{S}_0 = \left\{u\colon u_x = F(u)/x + \epsilon F \exp\left[(n-1)\int_0^u \frac{1}{F(z)}\,dz\right]\right\},$$

where $\epsilon \neq 0$ and $n \neq 1$ are two constants. If $\epsilon = 0$, the set reduces to the set S_0 introduced by Galaktionov [8], [9]. Finally, in Sec. 4, we introduce an extension of both S_0 and R_0, which is characterized by the invariant set $E_0 = \{u(x)\colon u_x = f(x)F(u),\ f' = af^2 + b\}$. Section 5 contains concluding remarks.

2. The extended rotation group

2.1. The rotation group. Equation (1) is invariant under the rotation group if it is invariant under the Lie group of transformations $x^* = x\cos\theta + u\sin\theta$, $u^* = -x\sin\theta + u\cos\theta$. This means that it admits the infinitesimal generator

$$V_0 = x\frac{\partial}{\partial u} - u\frac{\partial}{\partial x}.$$

If the equation admits the rotation group, the corresponding solution can be obtained by solving $u_x = -x/u$ together with Eq. (1). Indeed, the equation $u_x = -x/u$ is the invariant-surface condition corresponding to the rotation group. In other words, the set $\{u\colon u_x = -x/u\}$ is invariant under the rotation group. Using it, we can obtain an exact solution that has the form $u = (g(t) - x^2)^{1/2}$.

2.2. Algebraic differentiation. In the case where Eq. (1) does not admit the rotation group, we begin our discussion of invariant sets and exact solutions by introducing the set of functions

$$R_0 = \{u\colon u_x = xF(u)\}, \tag{2}$$

where $F(u)$ is a C^∞ function to be determined from the invariance condition

$$u(x,0) \in R_0 \Rightarrow u(x,t) \in R_0 \quad \text{for } t \in (0,1].$$

The structure of the manifold was used to construct the first-order sign invariants for radially symmetric parabolic equations (see p. 1605 in [6]). For $F = -1/u$, the contact structure characterizes the rotation group.

In the general case where

$$u_x = xF(u), \tag{3}$$

we list several derivatives that are used later:

$$\begin{aligned} u_{xx} &= x^2FF' + F,\\ u_{xxx} &= x^3F(FF')' + 3xFF',\\ u_{xxxx} &= x^4F(F(FF')')' + 6x^2F(FF')' + 3FF', \end{aligned} \tag{4}$$

where the prime denotes differentiation with respect to u. We now apply the approach to several typical nonlinear evolution equations.

2.3. Quasilinear heat equations. We consider a general reaction–diffusion equation of parabolic type

$$u_t = D(u)u_{xx} + Q(u)u_x^2 + G(u) \equiv A_1(u), \tag{5}$$

where $D \geq 0$, Q, and G are smooth functions. With formulas (4) substituted in (5), the right-hand side of (5) becomes

$$A_1(u) = x^2(DFF' + QF^2) + DF + G.$$

For $u \in R_0$, integrating $u_x = xF(u)$ implies that the solution is given by the implicit expression

$$\int^u \frac{1}{F(z)}\,dz = \frac{1}{2}x^2 + g(t), \tag{6}$$

where the function $g(t)$ is to be determined. It follows from (6) that

$$u_t = g'(t)F(u). \tag{7}$$

Therefore, Eq. (5) is equivalent to

$$g'(t) = x^2(DF' + QF) + D + \frac{G}{F}.$$

Because the left-hand side of this equation is independent of x, differentiating it with respect to x yields

$$x^3F(DF' + QF)' + x\left[F\left(D + \frac{G}{F}\right)' + 2(DF' + QF)\right] \equiv 0,$$

which implies that D, F, and G satisfy

$$\begin{aligned} &DF' + QF = c, \\ &F\left(D + \frac{G}{F}\right)' = -2c \Longrightarrow D + \frac{G}{F} = d - 2c\int^u \frac{1}{F(z)}\,dz, \end{aligned} \tag{8}$$

where c and d are arbitrary constants. Consequently, g satisfies the equation

$$g' = d - 2cg, \tag{9}$$

whose general solution is

$$g = \frac{d}{2c} + c_0e^{-2ct}, \tag{10}$$

where c_0 is another arbitrary constant.

System (8) can be easily solved if we know two of the four functions. We consider several special cases.

Case 1. $F = -1/u$, $D = u^m$. In this case, the invariant set R_0 characterizes the rotation group, namely, the equations are invariant under the rotation group. Solving system (8), we obtain

$$Q = -cu + u^{m-1}, \qquad G = u^{m-1} - cu - du^{-1}.$$

We have thus shown that the equation

$$u_t = u^m u_{xx} + (u^{m-1} - cu)u_x^2 + u^{m-1} - cu - du^{-1}$$

admits the rotation group and has the exact solution

$$u = \left(-\frac{d}{c} - 2c_0e^{-2ct} - x^2\right)^{1/2}. \tag{11}$$

Case 2. $F = u^k$, $D = u^m$, $k \neq -1$. In this case, $Q = cu^{-k} - ku^{m-1}$, and G satisfies

$$G = du^k - u^{m+k} - 2cu^k \int^u \frac{1}{F(z)}\, dz.$$

Two subcases are distinguished: $k = 1$ and $k \neq 1$.

Subcase 2a. $k \neq 1$. We obtain

$$G = du^k - u^{k+m} - \frac{2c}{1-k} u,$$

which means that the equation

$$u_t = u^m u_{xx} + (cu^{-k} - ku^{m-1})u_x^2 + du^k - u^{k+m} - \frac{2c}{1-k} u$$

has the solution

$$u = \left[\frac{1-k}{2}\left(x^2 + \frac{d}{c} + 2c_0 e^{-2ct}\right)\right]^{1/(1-k)}. \tag{12}$$

Subcase 2b. $k = 1$. We have

$$G = du - u^{m+1} - 2cu \log u.$$

The equation

$$u_t = u^m u_{xx} + (cu^{-1} - u^{m-1})u_x^2 + du - u^{m+1} - 2cu \log u$$

has the exact solution

$$u = \exp\left[\frac{x^2}{2} + \frac{d}{2c} + c_0 e^{-2ct}\right]. \tag{13}$$

Case 3. $F = u^k$, $D = e^u$. In this case,

$$Q = cu^{-k} - ku^{-1}e^u, \qquad G = du^k - u^k e - 2cu^k \int^u \frac{1}{F(z)}\, dz.$$

Two subcases arise.

Subcase 3a. $k \neq 1$. Here G is given by

$$G = du^k - u^k e^u + \frac{2c}{k-1} u.$$

The equation

$$u_t = e^u u_{xx} + (cu^{-k} - ku^{-1}e^u)u_x^2 + du^k - u^k e^u + \frac{2c}{k-1} u$$

has solution (12).

Subcase 3b. $k = 1$. Here G takes the form

$$G = du - ue^u - 2cu \log u.$$

The equation

$$u_t = e^u u_{xx} + (cu^{-k} - ku^{-1}e^u)u_x^2 + du - ue^u - 2cu \log u$$

has solution (13).

Case 4. $F = u^k$, $D = 1/(1+u^2)$. In this case,

$$Q = cu^{-k} - ku^{-1}(1+u^2)^{-1}, \qquad G = du^k - \frac{u^k}{1+u^2} - 2cu^k \int^u \frac{1}{F(z)}\,dz.$$

There are two subcases.

Subcase 4a. $k \neq 1$.

$$G = du^k - (1+u^2)^{-1}u^k - \frac{2c}{1-k}u.$$

Subcase 4b. $k = 1$.

$$G = du - u(1+u^2)^{-1} - 2cu\log u.$$

Therefore, the solutions of the equations

$$u_t = \frac{1}{1+u^2}u_{xx} + (cu^{-k} - ku^{-1}(1+u^2)^{-1})u_x^2 + du^k - (1+u^2)^{-1}u^k - \frac{2c}{1-k}u,$$

$$u_t = \frac{1}{1+u^2}u_{xx} + (cu^{-1} - u^{-1}(1+u^2)^{-1})u_x^2 + du - u(1+u^2)^{-1} - 2cu\log u$$

are given respectively by (12) and (13).

3. The extended scaling group

3.1. Invariant set. The Lie group of scaling transformations

$$x^* = e^{\epsilon}x, \qquad t^* = e^{\mu\epsilon}t,$$

where ϵ is a parameter, has the infinitesimal generator

$$X = x\frac{\partial}{\partial x} + \mu t\frac{\partial}{\partial t}.$$

If the function E in Eq. (1) is homogeneous, i.e., if it satisfies the condition

$$E\left(sx, u, \frac{1}{s}u_1, \dots, \frac{1}{s^m}u_m\right) = \left(\frac{1}{s}\right)^{\mu} A(x, u, u_1, \dots, u_m),$$

then Eq. (1) admits the self-similar solutions

$$u = \theta(\xi), \qquad \xi = \frac{x}{t^{1/\mu}}, \tag{14}$$

and the PDE is reduced to an ODE for the function θ,

$$E(\xi, \theta, \theta', \dots, \theta^m) + \frac{1}{\mu}\xi\theta' = 0.$$

Galaktionov [8], [9] proposed an interesting and natural generalization of the scaling group. He considered the equation in an invariant set $S_0 = \{u(x)\colon u_x = F(u)/x\}$, where F is a C^∞ function to be determined from the invariance condition

$$u(\,\cdot\,,0) \in S_0 \Rightarrow u(\,\cdot\,,t) \in S_0 \quad \text{for } t \in (0,1].$$

It is easy to see that the contact structure of the equation

$$u_x = \frac{1}{x}F(u)$$

includes scaling invariant (14).

For equations without scaling invariance, we introduce the invariant set

$$S_1 = \left\{u \in C^\infty,\ u_x = \frac{s}{x}F(u) + \epsilon F(u)\exp\left[(n-1)\int^u \frac{1}{F(z)}\,dz\right]\right\}, \tag{15}$$

where ϵ, s, and $n \neq 1$ are some constants. If $\epsilon = 0$, the set S_1 is reduced to the invariant set S_0. For simplicity, we consider only the case $F = u$ in the following applications, namely, the invariant set becomes

$$S_2 = \left\{u \in C^\infty,\ u_x = \frac{s}{x}u + u^n\right\}, \tag{16}$$

where ϵ has been set equal to one without loss of generality. In the set S_2, we have the formulas

$$\begin{aligned}
u_x &= \frac{s}{x}u + u^n,\\
u_{xx} &= \frac{1}{x^2}(s^2 - s)u + \frac{1}{x}s(n+1)u^n + nu^{2n-1},\\
u_{xxx} &= \frac{1}{x^3}s_1 u + \frac{1}{x^2}s_2 u^n + \frac{1}{x}s_3 u^{2n-1} + s_4 u^{3n-2},\\
u_{xxxx} &= \frac{1}{x^4}l_1 u + \frac{1}{x^3}l_2 u^n + \frac{1}{x^2}l_3 u^{2n-1} + \frac{1}{x}l_4 u^{3n-2} + l_5 u^{4n-3},
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
&s_1 = (s^2 - s)(s-2), \qquad s_2 = (n^2 + n + 1)s^2 - (n+2)s, \qquad s_3 = 3n^2 s,\\
&s_4 = n(2n-1), \qquad l_1 = (s-3)s_1, \qquad l_2 = s_1 + (ns-2)s_2,\\
&l_3 = ns_2 + [(2n-1)s - 1]s_3, \qquad l_4 = (2n-1)s_3 + (3n-2)ss_4,\\
&l_5 = (3n-2)s_4.
\end{aligned}$$

The solution of Eq. (1) in the invariant set S_2 is given by

$$u = \left[-\frac{(n-1)}{s(n-1)+1}x + g(t)x^{s(1-n)}\right]^{1/(1-n)}, \tag{18}$$

where $g(t)$ is to be determined. This type of solution has appeared in physical situations, such as a special "dipole" solution taking this form [10]. In what follows, we consider some applications of the invariant set S_2 to Eq. (1) and derive exact solutions of some parabolic and KdV-like equations.

3.2. Quasilinear heat equations. For the quasilinear heat equation of form (5), we can use the differentiation rules and compute that

$$\begin{aligned}
A_1(u) = \frac{1}{x^2}[(s^2 - s)uD + s^2u^2Q] + \frac{1}{x}[(n+1)su^nD + 2su^{n+1}Q] +\\
+ nu^{2n-1}D + u^{2n}Q + G(u)
\end{aligned}$$

on the set S_2. It follows from (18) that

$$u_t = \frac{1}{1-n} x^{-a} g'(t) u^n,$$

where $a = s(n-1)$. Hence, $g'(t)$ satisfies

$$\frac{1}{1-n} g'(t) = x^{a-2}[(s^2 - s)u^{1-n} D + s^2 u^{2-n} Q] +$$

$$+ x^{a-1}[(n+1)sD + 2suQ] + x^a[nu^{n-1}D + u^n Q + u^{-n} G] \equiv$$

$$\equiv x^{a-2} D_1 + x^{a-1} D_2 + x^a D_3. \tag{19}$$

Because the left-hand side of this equation is independent of x, differentiating it with respect to x yields

$$x^{a-3}[suD_1' + (a-2)D_1] + x^{a-2}[u^n D_1' + suD_2' + (a-1)D_2] +$$

$$+ x^{a-1}[u^n D_2' + suD_3' + aD_3] + x^a u^n D_3' = 0,$$

which gives

$$suD_1' + (a-2)D_1 = 0, \tag{20}$$

$$u^n D_1' + suD_2' + (a-1)D_2 = 0, \tag{21}$$

$$u^n D_2' + suD_3' + aD_3 = 0, \tag{22}$$

$$D_3' = 0. \tag{23}$$

Integrating (21)–(23) gives

$$D_3 = c_1,$$

$$D_2 = \frac{a}{n-1} c_1 u^{1-n} + c_2,$$

$$D_1 = -\frac{a}{2(n-1)^2} c_1 u^{2-2n} + \frac{a-1}{n-1} c_2 u^{1-n} + c_3,$$

where c_i, $i = 1, 2, 3$, are integration constants. Substituting these expressions for D_1, D_2, and D_3 in (20) implies

$$(a-2)c_3 = 0,$$

$$(a-1)c_2 = 0, \tag{24}$$

$$(a+2)c_1 = 0.$$

The case $a = -2$ yields the trivial solution $D = Q = G = 0$. Therefore, two possibilities arise.

Case 1. $a = 1$, $c_3 = c_1 = 0$, $c_2 \neq 0$. In this case, D, Q, and G satisfy

$$(s^2 - s)u^{1-n} D + s^2 u^{2-n} Q = 0,$$

$$(n+1)sD + 2suQ = c_2,$$

$$nu^{n-1} D + u^n Q + u^{-n} G = 0,$$

which has the solution

$$D = \frac{1}{3}c_2,$$

$$Q = \frac{n-2}{3u}c_2,$$

$$G = -\frac{2(n-1)u^{2n-1}}{3}c_2.$$

In this case, $g(t)$ satisfies

$$g' = (1-n)c_2,$$

which gives

$$g(t) = (1-n)c_2(t-t_0).$$

This means that the equation

$$u_t = \frac{c_2}{3}\left[u_{xx} + (n-2)\frac{u_x^2}{u} - 2(n-1)u^{2n-1}\right]$$

has the solution

$$u = \left[(1-n)\left(\frac{x}{2} + c_2\frac{t-t_0}{x}\right)\right]^{1/(1-n)}. \tag{25}$$

Case 2. $a = 2$, $c_1 = c_2 = 0$, $c_3 \neq 0$. In this case, D, Q, and G satisfy

$$(s^2 - s)u^{1-n}D + s^2u^{2-n}Q = c_3,$$

$$(n+1)sD + 2suQ = 0,$$

$$nu^{n-1}D + u^nQ + u^{-n}G = 0,$$

which gives

$$D = -\frac{n-1}{4}u^{n-1}c_3,$$

$$Q = \frac{n^2-1}{8}u^{n-2}c_3,$$

$$G = \frac{(n-1)^2}{8}u^{3n-2}c_3.$$

In this case, $g(t)$ is given by

$$g = (1-n)c_3(t-t_0).$$

Therefore, the solution of

$$u_t = \frac{c_3(n-1)}{4}u^{n-1}\left[-u_{xx} + \frac{(n+1)}{2}\frac{u_x^2}{u} + \frac{(n-1)}{2}u^{2n-1}\right]$$

is

$$u = \left[(1-n)\left(\frac{x}{3} + c_3\frac{t-t_0}{x^2}\right)\right]^{1/(1-n)}. \tag{26}$$

4. The extended scaling–rotation group

In this section, we propose an extension for both the extended scaling group (S_0) and the rotation group (R_0), which is characterized by the invariant set

$$E_0 = \{u\colon u_x = f(x)F(u)\}, \tag{27}$$

where $f(x)$ satisfies

$$f_x = af^2 + b. \tag{28}$$

Therefore,

$$f = \frac{\sqrt{ab}\tan(\sqrt{ab}(x+x_0))}{a} \quad \text{if } ab > 0$$

or

$$f = \frac{\sqrt{-ab}\coth(\sqrt{-ab}(x+x_0))}{a},$$
$$f = -\frac{\sqrt{-ab}\tanh(\sqrt{-ab}(x+x_0))}{a} \quad \text{if } ab < 0,$$

where $a, b \neq 0$ are two constants. We note that when $a = 0$ or $b = 0$, the invariant set E_0 respectively reduces to R_0 or S_0. Therefore, the invariant set E_0 can be considered a generalization of S_0 and R_0. In the invariant set E_0, we have the formulas

$$\begin{aligned}
u_x &= f(x)F(u),\\
u_{xx} &= f^2(FF' + aF) + bF,\\
u_{xxx} &= f^3[F(FF' + aF)' + 2aF(F' + a)] + bfF(3F' + 2a),\\
u_{xxxx} &= f^4\{F[F(FF' + aF)' + 2aF(F' + a)]' + 3aF[(FF' + aF)' + 2a(F' + a)]\} +\\
&\quad + f^2\{3b[F(FF' + aF)' + 2aF(F' + a)] + bF(3FF' + 2aF)' +\\
&\quad + abF(3F' + 2a)\} + b^2F(3F' + 2a).
\end{aligned} \tag{29}$$

In the invariant set E_0, the solution takes the form

$$\int^u \frac{ds}{F(s)} = -\frac{1}{a}\log|\cos(\sqrt{ab}(x+x_0))| + g(t) \quad \text{if } ab > 0 \tag{30}$$

or

$$\begin{aligned}
\int^u \frac{ds}{F(s)} &= -\frac{1}{a}\log|\sinh(\sqrt{-ab}(x+x_0))| + g(t) \quad \text{if } ab < 0,\\
\int^u \frac{ds}{F(s)} &= -\frac{1}{a}\log|\cosh(\sqrt{-ab}(x+x_0))| + g(t) \quad \text{if } ab < 0.
\end{aligned} \tag{31}$$

We now use the above formulas to construct exact solutions of the following nonlinear evolution equations.

Quasilinear heat equations. We consider the quasilinear heat equation. Substituting (29) in (5), we have

$$g'(t) = f^2(DF' + QF + aD) + \frac{G}{F} + bD. \tag{32}$$

The left-hand side of (33) is independent of x. Therefore, we have

$$(DF' + QF + aD)'F + 2a(DF' + QF + aD) = 0,$$

$$\left(\frac{G}{F} + bD\right)' F + 2b(DF' + QF + aD) = 0.$$

Integrating gives

$$DF' + QF + aD = c_0 \exp\left[-2a \int^u \frac{ds}{F(s)}\right],$$

$$\frac{G}{F} + bD = \frac{bc_0}{a} \exp\left[-2a \int^u \frac{ds}{F(s)}\right] + c_1,$$

where c_0 and c_1 are arbitrary constants. Hence, $g(t)$ satisfies the equation

$$g' = c_1 + \frac{b}{a} c_0 e^{-2ag} \Longrightarrow g = \frac{1}{2a} \log\left[\frac{bc_0(e^{2ac_1(t+t_0)} - 1)}{ac_1}\right].$$

In particular, we find that the equation

$$u_t = u_{xx} + G(u)$$

admits solutions (30), (31) when F satisfies

$$FF'' + 2a(F' + a) = 0$$

and $G(u)$ is given by

$$G = -2bF \log F - 2abF \int^u \frac{ds}{F(s)} + cF,$$

where c is a constant.

Remark. This approach can also be used for equations with x-variable coefficients. For example, the equation

$$u_t = u_{xx} + X(x)G(u)$$

also admits solutions (30), (31), where F, G, and X satisfy

$$F\left(F' + \alpha\frac{G}{F}\right)' + 2a\left(F' + a + \alpha\frac{G}{F}\right) = 0,$$

$$\beta\left(\frac{G}{F}\right)' F + 2b\left(F' + \alpha\frac{G}{F} + a\right) = 0,$$

$$X(x) = \alpha f^2 + \beta,$$

where α and β are constants.

5. Conclusions

We have shown that the rotation group admits a "nonlinear" extension characterized by an invariant set of a contact first-order differential structure, which is parallel to the "nonlinear" extension for the scaling groups. This extension has been used to construct exact solutions of some nonlinear evolution equations of the second and fourth orders that do not admit the rotation group. We have proposed a more general "nonlinear" extension of the scaling group that is also characterized by an invariant set of a contact first-order differential structure with a nonhomogeneous term. When the nonhomogeneous term is zero, our set reduces to the invariant set introduced in [8]. A further extension of both the extended scaling group S_0 and the extended rotation group R_0 was also introduced. These approaches were then used to obtain some exact solutions of nonlinear evolution equations including KdV-type equations.

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