



Lie symmetries for Lie systems: Applications to systems of ODEs and PDEs



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ABSTRACT

A Lie system is a nonautonomous system of first-order differential equations admitting a *superposition rule*, i.e., a map expressing its general solution in terms of a generic family of particular solutions and some constants. Using that a Lie system can be considered as a curve in a finite-dimensional Lie algebra of vector fields, a so-called *Vessiot–Guldberg Lie algebra*, we associate every Lie system with a Lie algebra of Lie point symmetries induced by the Vessiot–Guldberg Lie algebra. This enables us to derive Lie symmetries of relevant physical systems described by first- and higher-order systems of differential equations by means of Lie systems in an easier way than by standard methods. A generalization of our results to partial differential equations is introduced. Among other applications, Lie symmetries for several new and known generalizations of the real Riccati equation are studied.

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1. Introduction

The analysis of Lie systems dates back to the end of the XIXth century, when Vessiot, Guldberg and Lie [22,25,33] discovered the most fundamental properties of nonautonomous systems of first-order ordinary differential equations (ODEs) admitting a superposition rule. Lie proved the nowadays called *Lie–Scheffers Theorem*, which states that a nonautonomous system of first-order ODEs admits a superposition rule if and only if it describes the integral curves of a t -dependent vector field taking values in a finite-dimensional real Lie algebra V of vector fields [9]. We call V a *Vessiot–Guldberg Lie algebra* of the Lie system [12].

The theory of Lie systems was deeply analyzed towards the end of the XIXth century, to be almost forgotten soon after. Along the eighties of the XXth century, Winternitz and coworkers derived superposition rules for many relevant systems of differential and superdifferential equations and studied the classification of Lie systems on manifolds among other topics [4,35].

Cariñena, Grabowski and Marmo characterized superposition rules as connections of zero curvature [9]. Other relevant results have also been derived for Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to some geometric structure [7,13]. This fact led to the derivation of a superposition rule for Riccati equations in a simpler way by using a Casimir function of $C^\infty(\mathfrak{sl}(2, \mathbb{R})^*)$ [3]. These results illustrate that Lie systems enjoy relevant geometric properties. Additionally, Lie systems can be applied in quantum/classical mechanics and control theory (see [12] and references therein).

The interest of Lie symmetries for the study of systems of differential equations is undoubtable. For instance, Lie symmetries allow us to write a differential equation in a simpler way. In this paper, we pioneer the study of Lie symmetries for Lie systems

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and their generalizations to higher-order ODEs (HODEs) and partial differential equations (PDEs): the *HODE and PDE Lie systems* [9,10]. As far as we know, only a few minor results about Lie symmetries for Lie systems had appeared before in [13].

Given a Lie system possessing a Vessiot–Guldberg Lie algebra V , we study its Lie symmetries of the form

$$Y = f_0(t) \frac{\partial}{\partial t} + \sum_{\alpha=1}^r f_\alpha(t) X_\alpha,$$

where f_0, \dots, f_r are certain t -dependent functions and X_1, \dots, X_r form a basis for V . We prove that f_0, \dots, f_r depend on the algebraic structure of V and can be derived by solving a Lie system, the symmetry system of the Lie system under study, which can be expressed in coordinates in a simple way. This enables us to calculate Lie symmetries for Lie systems with isomorphic Vessiot–Guldberg Lie algebras simultaneously.

As an application, we study Lie symmetries for Riccati equations. We retrieve and generalize results given in [21] to determine their Lie symmetries. Subsequently, we propose their generalization to the realm of split-complex and Study numbers [32,36]: the *Cayley–Klein Riccati equation*. We prove that the Cayley–Klein Riccati equation with real t -dependent coefficients is a Lie system admitting a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, namely an $\mathfrak{sl}(2, \mathbb{R})$ -Lie system [30], and analyze its Lie symmetries with our methods. Next, we study quaternionic Riccati equations with real t -dependent coefficients. We prove that they are $\mathfrak{sl}(2, \mathbb{R})$ -Lie systems and give a Lie system describing some of its Lie symmetries. Since all of the above mentioned Lie systems are $\mathfrak{sl}(2, \mathbb{R})$ -Lie systems, our techniques enable us to obtain a common equation describing Lie symmetries for all of them.

As a particular example, we generalize results concerning Lie symmetries for generalized and classical Darboux–Brioschi–Halphen systems [28]. Apart from the interest in these systems due to their applications [14], our example illustrates how Lie systems can be employed to investigate autonomous systems for the first time. Autonomous systems of first-order ordinary differential equations are Lie systems associated with a one-dimensional Vessiot–Guldberg Lie algebra. This implies that most of the methods to analyze Lie systems cannot be applied, as they only work properly for Vessiot–Guldberg Lie algebras of dimension two or higher (see [12]). We show that this problem can be avoided by studying autonomous systems as Lie systems related to Vessiot–Guldberg Lie algebras of dimension greater than one. Although finding such Lie algebras can be difficult, we succeeded in determining some of them for certain relevant Lie systems.

Subsequently, we illustrate how our results can be used to study HODE Lie systems [8]. Such systems are systems of higher-order differential equations that can be written as a Lie system by considering them as a first-order system in the usual way, namely by adding extra variables $v \equiv dx/dt$, $a \equiv dv/dt$, etc. Using this fact and the techniques of this work, we obtain *non-local Lie symmetries* for the initial HODE Lie system under study, where by non-local we mean Lie symmetries depending on the successive derivatives of the dependent variables. We also study Lie algebras of Lie point symmetries for HODE Lie systems, i.e., Lie symmetries that only depend on their independent and dependent variables.

Finally, we extend our results to PDE Lie systems [9]. Similarly to Lie systems, PDE Lie systems can be associated with a Vessiot–Guldberg Lie algebra. Also, they can be related to a Lie algebra of Lie symmetries of a certain type which is again determined by the Vessiot–Guldberg Lie algebra of the initial system. As before, this permits us to calculate simultaneously Lie symmetries for all PDE Lie systems possessing isomorphic Vessiot–Guldberg Lie algebras by solving another PDE Lie system. To illustrate our results, we analyze PDE Lie systems of relevance in physics and mathematics, e.g., multidimensional Riccati equations and PDEs describing certain flat connection forms [20]. It is remarkable that the literature on PDE Lie systems is scarce (specially concerning their uses) and this paper contributes to enlarge their applications and to understand their properties [12].

The structure of the paper goes as follows. In Section 2, we briefly review the concept of Lie systems and PDE Lie systems. Section 3 is devoted to the analysis of Lie symmetries for Lie systems. In Section 4 we provide methods to build different types of Lie algebras of Lie symmetries for Lie systems. Subsequently, we apply our results to Lie systems and HODE Lie systems of relevance in Section 5. We extend our methods to PDE Lie systems in Sections 6 and 7. In particular, we calculate Lie symmetries for PDE Lie systems with the same symmetry system and physical and mathematical interest. In Section 8, we summarize our results and we describe our plans for future research.

2. Fundamentals on Lie systems

For simplicity, we hereafter assume all geometric structures to be real, smooth and globally defined. In this way, we highlight the key points of our presentation by omitting the analysis of minor technical problems (we refer to [9,12] for additional details). Subsequently, N is assumed to be a real n -dimensional manifold with a local coordinate system $\{x_1, \dots, x_n\}$.

Let $\tau: TN \rightarrow N$ be the natural projection of the tangent bundle TN onto N and let $\pi_2: (t, x) \in \mathbb{R} \times N \mapsto x \in N$ be a projection. A t -dependent vector field X on N is a map $X: (t, x) \in \mathbb{R} \times N \mapsto X(t, x) \in TN$ such that $\tau \circ X = \pi_2$. This condition implies that every t -dependent vector field X on N can be considered as a family $\{X_t\}_{t \in \mathbb{R}}$ of vector fields $X_t: x \in N \mapsto X_t(x) \equiv X(t, x) \in T_x N \subset TN$ on N and vice versa. We call $\Gamma(TN)$ the $C^\infty(N)$ -module of sections of the tangent bundle (TN, N, τ) .

We call *integral curve* of X an integral curve $\gamma: s \in \mathbb{R} \mapsto (t(s), x(s)) \in \mathbb{R} \times N$ of the *suspension* of X , i.e., the vector field

$$\begin{aligned} \bar{X}: \mathbb{R} \times N &\longrightarrow T(\mathbb{R} \times N) \simeq T\mathbb{R} \oplus TN \\ (t, x) &\mapsto \frac{\partial}{\partial t} + X(t, x). \end{aligned} \tag{2.1}$$

The integral curves of X of the form $(t, x(t))$ are particular solutions of the system

$$\frac{dx_i}{dt} = X_i(t, x), \quad i = 1, \dots, n. \tag{2.2}$$

This is the so-called *associated system* with X . Conversely, a system of this type describes the integral curves of the form $(t, x(t))$ of a unique t -dependent vector field on N , namely,

$$X(t, x) = \sum_{i=1}^n X_i(t, x) \frac{\partial}{\partial x_i}. \tag{2.3}$$

This justifies to employ the symbol X to refer to both a t -dependent vector field and its associated system.

Definition 2.1. A *superposition rule* for a system X on N is a function $\Phi: N^m \times N \rightarrow N$ of the form $x = \Phi(x_{(1)}, \dots, x_{(m)}; k)$ satisfying that the general solution $x(t)$ of X can be written as

$$x(t) = \Phi(x_{(1)}(t), \dots, x_{(m)}(t); k), \tag{2.4}$$

where $x_{(1)}(t), \dots, x_{(m)}(t)$ is any generic family of particular solutions to X and $N \ni k = (k_1, \dots, k_n)$ can be understood as a set of constants to be related to initial conditions of X .

Theorem 2.2. (The Lie–Scheffers Theorem) A system X on N admits a superposition rule if and only if

$$X(t, x) = \sum_{\alpha=1}^r b_\alpha(t) X_\alpha(x) \tag{2.5}$$

for certain t -dependent functions b_1, \dots, b_r and vector fields X_1, \dots, X_r on N spanning an r -dimensional real Lie algebra: a *Vessiot–Guldberg Lie algebra* of X .

If X is a Lie system admitting a Vessiot–Guldberg Lie algebra isomorphic to a Lie algebra \mathfrak{g} , we call X a \mathfrak{g} -Lie system [30].

Lie systems and superposition rules can be generalized to PDEs as follows [9]. Consider a system of PDEs

$$\frac{\partial x_i}{\partial t_l} = X_{li}(t, x), \quad i = 1, \dots, n, \quad l = 1, \dots, s, \tag{2.6}$$

where the $X_{li}: (t, x) \in \mathbb{R}^s \times \mathbb{R}^n \mapsto X_{li}(t, x) \in \mathbb{R}$ are arbitrary functions. Its particular solutions give rise to sections $\sigma: t \mapsto (t_1, \dots, t_s) \in \mathbb{R}^s \mapsto (t, x(t)) \in \mathbb{R}^s \times N$ of the bundle $(\mathbb{R}^s \times N, \mathbb{R}^s, \pi_s: (t, x) \in \mathbb{R}^s \times N \mapsto t \in \mathbb{R}^s)$. In particular, we recover the simple case of ODEs when $s = 1$.

Let us assume $t \in \mathbb{R}^s$. We call t -dependent vector field X on N a map $X: \mathbb{R}^s \times N \rightarrow TN$ such that $\tau \circ X = \pi_s$. We call \tilde{X}^l , with $l = 1, \dots, s$, the *autonomization* of X with respect to t_l , i.e.,

$$\tilde{X}^l(t, x) \equiv \frac{\partial}{\partial t_l} + X(t, x). \tag{2.7}$$

Definition 2.3. A system of PDEs (2.6) is a *PDE Lie system* if:

- There exist vector fields X_1, \dots, X_r on N spanning an r -dimensional real Lie algebra V satisfying that each t -dependent vector field $X_l(t, x) = \sum_{i=1}^n X_{li}(t, x) \partial/\partial x_i$, where $l = 1, \dots, s$, can be written in the form

$$X_l(t, x) = \sum_{\alpha=1}^r b_{\alpha l}(t) X_\alpha(x), \quad l = 1, \dots, s$$

for certain t -dependent functions $b_{\alpha l}$.

- Let $c_{\alpha\beta\gamma}$ be the structure constants of X_1, \dots, X_r , i.e., $[X_\alpha, X_\beta] = \sum_{\gamma=1}^r c_{\alpha\beta\gamma} X_\gamma$. Then,

$$\frac{\partial b_{\gamma k}}{\partial t_l}(t) - \frac{\partial b_{\gamma l}}{\partial t_k}(t) + \sum_{\alpha, \beta=1}^r b_{\alpha l}(t) b_{\beta k}(t) c_{\alpha\beta\gamma} = 0, \quad l \neq k = 1, \dots, s, \quad \gamma = 1, \dots, r.$$

If these conditions are satisfied, we call V a *Vessiot–Guldberg Lie algebra for the PDE Lie system* (2.6).

3. On certain Lie symmetries for Lie systems

We are now concerned with the study of certain Lie symmetries for Lie systems. We prove that the features of these Lie symmetries are determined by the algebraic structure of a Vessiot–Guldberg Lie algebra of the Lie system.

Consider again the Lie system X given by (2.2) admitting a Vessiot–Guldberg Lie algebra V with basis X_1, \dots, X_r and structure constants $c_{\alpha\beta\gamma}$. Then, X can be brought into the form (2.5) for some t -dependent functions b_1, \dots, b_r (with $t \in \mathbb{R}$). Let us study its Lie symmetries of the form

$$Y = f_0(t) \frac{\partial}{\partial t} + \sum_{\alpha=1}^r f_\alpha(t) X_\alpha, \tag{3.1}$$

where f_0, \dots, f_r are certain t -dependent functions. We denote by S_X^V the space of such Lie symmetries. Recall that $Y \in S_X^V$ if and only if

$$[Y, \bar{X}] = h\bar{X} \tag{3.2}$$

for a function $h \in C^\infty(\mathbb{R} \times N)$, with \bar{X} being the autonomization of X given by (2.1) [29]. From this, it follows that S_X^V is a real Lie algebra of vector fields.

Using properties of S_X^V above, we now characterize the elements of S_X^V as particular solutions of a family of Lie systems.

Lemma 3.1. *The vector field Y of the form (3.1) is a Lie symmetry for the Lie system (2.5) if and only if the t -dependent functions f_0, \dots, f_r satisfy the system of differential equations*

$$\frac{df_0}{dt} = b_0(t), \quad \frac{df_\alpha}{dt} = f_0 \frac{db_\alpha}{dt}(t) + b_\alpha(t)b_0(t) + \sum_{\beta,\gamma=1}^r b_\beta(t)f_\gamma c_{\gamma\beta\alpha}, \quad \alpha = 1, \dots, r, \tag{3.3}$$

for a certain t -dependent function b_0 .

Proof. From (2.5), (3.1) and (3.2), we have that

$$\begin{aligned} [Y, \bar{X}] &= \left[f_0 \frac{\partial}{\partial t} + \sum_{\alpha=1}^r f_\alpha X_\alpha, \frac{\partial}{\partial t} + \sum_{\beta=1}^r b_\beta X_\beta \right] \\ &= -\frac{df_0}{dt} \frac{\partial}{\partial t} + \sum_{\alpha=1}^r \left(\left(f_0 \frac{db_\alpha}{dt} - \frac{df_\alpha}{dt} \right) X_\alpha + \sum_{\beta=1}^r b_\beta f_\alpha [X_\alpha, X_\beta] \right) \\ &= -\frac{df_0}{dt} \frac{\partial}{\partial t} + \sum_{\alpha=1}^r \left(\left(f_0 \frac{db_\alpha}{dt} - \frac{df_\alpha}{dt} \right) X_\alpha + \sum_{\beta,\gamma=1}^r b_\beta f_\alpha c_{\alpha\beta\gamma} X_\gamma \right) \\ &= -\frac{df_0}{dt} \frac{\partial}{\partial t} + \sum_{\alpha=1}^r \left(f_0 \frac{db_\alpha}{dt} - \frac{df_\alpha}{dt} + \sum_{\beta,\gamma=1}^r b_\beta f_\gamma c_{\gamma\beta\alpha} \right) X_\alpha \\ &= h \left(\frac{\partial}{\partial t} + \sum_{\alpha=1}^r b_\alpha X_\alpha \right). \end{aligned}$$

Thus, $[Y, \bar{X}] = h\bar{X}$ is equivalent to

$$\left(-\frac{df_0}{dt} - h \right) \frac{\partial}{\partial t} + \sum_{\alpha=1}^r \left[\sum_{\beta,\gamma=1}^r b_\beta f_\gamma c_{\gamma\beta\alpha} + f_0 \frac{db_\alpha}{dt} - \frac{df_\alpha}{dt} - hb_\alpha \right] X_\alpha = 0.$$

Since $\partial/\partial t, X_1, \dots, X_r$, are linearly independent over \mathbb{R} , we obtain that $[Y, \bar{X}] = h\bar{X}$ if and only if (3.3) is fulfilled with h being an arbitrary t -dependent function such that $b_0(t) = -h(t)$ for every $t \in \mathbb{R}$. Hence, the elements of S_X^V are the particular solutions of (3.3) for a t -dependent function b_0 . \square

Definition 3.2. We call (3.3) the *symmetry system* of the Lie system (2.5) with respect to its Vessiot–Guldberg Lie algebra V . We write Γ_X^V for the t -dependent vector field associated with (3.3), that is,

$$\Gamma_X^V = b_0(t) \frac{\partial}{\partial f_0} + \sum_{\alpha=1}^r \left(f_0 \frac{db_\alpha}{dt}(t) + b_0(t)b_\alpha(t) + \sum_{\gamma,\beta=1}^r b_\beta(t)f_\gamma c_{\gamma\beta\alpha} \right) \frac{\partial}{\partial f_\alpha}.$$

Note that (2.5) may have different Vessiot–Guldberg Lie algebras (see [12] for details). When V is understood from context, we simply call (3.3) the symmetry system of X . Let us prove that (3.3) is a Lie system.

Theorem 3.3. *The system Γ_X^V is a Lie system possessing a Vessiot–Guldberg Lie algebra*

$$(A_1 \oplus_S A_2) \oplus_S V_L \simeq (\mathbb{R}^{r+1} \oplus_S \mathbb{R}^r) \oplus_S V/Z(V), \tag{3.4}$$

where

$$A_1 = \langle Z_0, \dots, Z_r \rangle \simeq \mathbb{R}^{r+1}, \quad A_2 = \langle W_1, \dots, W_r \rangle \simeq \mathbb{R}^r, \quad V_L = \langle Y_1, \dots, Y_r \rangle \simeq V/Z(V),$$

with

$$Y_\alpha = \sum_{\beta,\gamma=1}^r f_\beta c_{\beta\alpha\gamma} \frac{\partial}{\partial f_\gamma}, \quad W_\alpha = f_0 \frac{\partial}{\partial f_\alpha}, \quad Z_0 = \frac{\partial}{\partial f_0}, \quad Z_\alpha = \frac{\partial}{\partial f_\alpha}, \quad \alpha = 1, \dots, r,$$

we write $A \oplus_S B$ for the semi-direct sum of the ideal A of $A + B$ with B , and $Z(V)$ is the center of the Lie algebra V .

Proof. By defining t -dependent functions

$$c_0 \equiv b_0, \quad \bar{b}_0 \equiv 0, \quad c_\alpha \equiv b_0 b_\alpha, \quad \bar{b}_\alpha \equiv \frac{db_\alpha}{dt}, \quad \alpha = 1, \dots, r,$$

we can write

$$\Gamma_X^V(t, f) = \sum_{\alpha=0}^r c_\alpha(t) Z_\alpha(f) + \sum_{\alpha=1}^r [\bar{b}_\alpha(t) W_\alpha(f) + b_\alpha(t) Y_\alpha(f)], \quad f \in \mathbb{R}^{n+1}.$$

Hence, Γ_X^V is a t -dependent vector field taking values in the linear space $V^S \equiv A_1 + A_2 + V_L$. Let us show that V^S is also a Lie algebra of vector fields. To do so, let us first prove that $V_L \simeq V/Z(V)$. Consider $Y_\alpha, Y_\beta \in V_L$. Recalling that $[X_\alpha, X_\beta] = \sum_{\gamma=1}^r c_{\alpha\beta\gamma} X_\gamma$, we obtain

$$\begin{aligned} [Y_\alpha, Y_\beta] &= \sum_{i,j,m,n=1}^r \left[c_{i\alpha j} f_i \frac{\partial}{\partial f_j}, c_{m\beta n} f_m \frac{\partial}{\partial f_n} \right] = \sum_{i,j,m,n=1}^r c_{i\alpha j} c_{m\beta n} \left(f_i \delta_j^m \frac{\partial}{\partial x_n} - f_m \delta_n^i \frac{\partial}{\partial f_j} \right) \\ &= \sum_{i,m,n=1}^r c_{i\alpha m} c_{m\beta n} f_i \frac{\partial}{\partial f_n} - \sum_{n,j,m=1}^r c_{n\alpha j} c_{m\beta n} f_m \frac{\partial}{\partial f_j} = \sum_{i,m,n=1}^r (c_{i\alpha m} c_{m\beta n} + c_{m\alpha n} c_{\beta im}) f_i \frac{\partial}{\partial f_n}. \end{aligned}$$

Using the Jacobi identity for the structure constants $c_{\alpha\beta\gamma}$, we see that

$$\sum_{m,n=1}^r (c_{i\alpha m} c_{m\beta n} + c_{\alpha\beta m} c_{min} + c_{\beta im} c_{m\alpha n}) = 0, \quad \forall i, \alpha, \beta = 1, \dots, r.$$

From this,

$$[Y_\alpha, Y_\beta] = \sum_{i,m,n=1}^r (c_{i\alpha m} c_{m\beta n} + c_{m\alpha n} c_{\beta im}) f_i \frac{\partial}{\partial f_n} = - \sum_{i,m,n=1}^r c_{\alpha\beta m} c_{min} f_i \frac{\partial}{\partial f_n} = \sum_{m=1}^r c_{\alpha\beta m} Y_m.$$

So, Y_1, \dots, Y_r span a Lie algebra. We can define a Lie algebra morphism $\phi: V \rightarrow V_L$ of the form $\phi(X_\alpha) = Y_\alpha$ for $\alpha = 1, \dots, r$. The vector fields Y_1, \dots, Y_r do not need to be linearly independent. Let us show this. We can assume with no loss of generality that X_1, \dots, X_s , with $s \leq r$, form a basis for $\ker \phi$. Since $\phi(X_\alpha) = 0$ for $\alpha = 1, \dots, s$, we have $f_\beta c_{\beta\alpha\gamma} = 0$ for $\alpha = 1, \dots, s$ and $\beta, \gamma = 1, \dots, r$. Thus, we see that $[X_\alpha, X_\beta] = 0$ for $\alpha = 1, \dots, s$ and $\beta = 1, \dots, r$. This means that $X_\alpha \in Z(V)$. Conversely, we get by similar arguments that if $X \in Z(V)$, then $X \in \ker \phi$. Hence, $X \in \ker \phi$ if and only if $X \in Z(V)$. In consequence, $\ker \phi = Z(V)$ and Y_1, \dots, Y_r span a Lie algebra isomorphic to $V/Z(V)$.

It is obvious that A_1 is an ideal of $A_1 + A_2$. Moreover, as $[A_1, V_L] \subset A_1$ and $[A_2, V_L] \subset A_2$, then $A_1 \oplus_S A_2$ is an ideal of V^S . Consequently, V^S is a Lie algebra of the form (3.4). \square

Definition 3.4. We say that the Lie systems X_1 and X_2 are isomorphic when they take values in two isomorphic Vessiot–Guldberg Lie algebras V_1, V_2 and there exists a Lie algebra isomorphism $\phi: V_1 \rightarrow V_2$ such that $(X_2)_t = \phi((X_1)_t)$ for each $t \in \mathbb{R}$.

Proposition 3.5. Given two isomorphic Lie systems X_1 and X_2 related to Vessiot–Guldberg Lie algebras V_1, V_2 , their symmetry systems relative to such Lie algebras are, up to a change of basis in V_1 and/or V_2 , the same.

4. Lie algebras of Lie symmetries for Lie systems

In this section we study different Lie subalgebras of S_X^V . Their interest resides in the fact that, when finite-dimensional, they can be integrated to form Lie group actions of symmetries for X that can be employed to simplify the Lie system they are referred to.

Lemma 4.1. The space of functions $C^\infty(\mathbb{R})$ can be endowed with a Lie bracket given by

$$\{f, f_*\}_{\mathbb{R}} = f \frac{df_*}{dt} - f_* \frac{df}{dt}, \quad \forall f, f_* \in C^\infty(\mathbb{R}). \tag{4.1}$$

Proof. In order to prove that (4.1) is a Lie bracket, we must show that (4.1) is bilinear, antisymmetric and satisfies the Jacobi identity. From its definition (4.1) is clearly bilinear and antisymmetric. To see that (4.1) holds the Jacobi identity, we consider the map $\phi: f \in C^\infty(\mathbb{R}) \mapsto f \partial / \partial t \in \Gamma(\mathbb{T}\mathbb{R})$. Observe that ϕ is a linear isomorphism.

Moreover,

$$\phi(\{f, f_*\}_{\mathbb{R}}) = \{f, f_*\}_{\mathbb{R}} \frac{\partial}{\partial t} = \left(f \frac{df_*}{dt} - f_* \frac{df}{dt} \right) \frac{\partial}{\partial t} = \left[f \frac{\partial}{\partial t}, f_* \frac{\partial}{\partial t} \right] = [\phi(f), \phi(f_*)], \quad \forall f, f_* \in C^\infty(\mathbb{R}).$$

By using the Jacobi identity for vector fields on \mathbb{R} with respect to the Lie bracket $[\cdot, \cdot]$, we obtain

$$\phi(\{ \{f, f_*\}_{\mathbb{R}}, f_{**} \}_{\mathbb{R}} + \{ \{f_*, f_{**}\}_{\mathbb{R}}, f \}_{\mathbb{R}} + \{ \{f_{**}, f\}_{\mathbb{R}}, f_* \}_{\mathbb{R}})$$

$$= [[\phi(f), \phi(f_*)], \phi(f_{**})] + [[\phi(f_*), \phi(f_{**})], \phi(f)] + [[\phi(f_{**}), \phi(f)], \phi(f_*)] = 0, \quad \forall f, f_*, f_{**} \in C^\infty(\mathbb{R}).$$

Since ϕ is a linear isomorphism,

$$\{\{f, f_*\}_{\mathbb{R}}, f_{**}\}_{\mathbb{R}} + \{\{f_*, f_{**}\}_{\mathbb{R}}, f\}_{\mathbb{R}} + \{\{f_{**}, f\}_{\mathbb{R}}, f_*\}_{\mathbb{R}} = 0$$

and (4.1) satisfies the Jacobi identity giving rise to a Lie bracket on $C^\infty(\mathbb{R})$. Moreover, ϕ becomes a Lie algebra isomorphism. \square

Definition 4.2. Let X be a Lie system on N with a Vessiot–Guldberg Lie algebra V and let \mathfrak{W} be a nonempty set of t -dependent functions that form a Lie algebra with respect to the Lie bracket defined in (4.1). We call $S_{X, \mathfrak{W}}^V$ the space

$$S_{X, \mathfrak{W}}^V = \left\{ Y \in S_X^V \mid Y = f_0 \frac{\partial}{\partial t} + \sum_{\alpha=1}^r f_\alpha X_\alpha, f_0 \in \mathfrak{W} \right\},$$

where X_1, \dots, X_r is a basis for V .

Proposition 4.3. The space of symmetries $S_{X, \mathfrak{W}}^V$ is a Lie algebra of symmetries of X .

Proof. Since \mathfrak{W} and S_X^V are linear spaces, the linear combinations of elements of $S_{X, \mathfrak{W}}^V$ belong to $S_{X, \mathfrak{W}}^V$. So, this space becomes a vector space. Moreover, given two elements, $Y, Y^* \in S_{X, \mathfrak{W}}^V$, their Lie bracket reads

$$\begin{aligned} [Y, Y^*] &= \left[f_0(t) \frac{\partial}{\partial t} + \sum_{\alpha=1}^r f_\alpha(t) X_\alpha, f_0^*(t) \frac{\partial}{\partial t} + \sum_{\beta=1}^r f_\beta^*(t) X_\beta \right] \\ &= \{f_0, f_0^*\}_{\mathbb{R}} \frac{\partial}{\partial t} + \sum_{\beta=1}^r \left[\left(f_0 \frac{df_\beta^*}{dt} - f_0^* \frac{df_\beta}{dt} \right) X_\beta + \sum_{\alpha, \gamma=1}^r f_\alpha f_\beta^* c_{\alpha\beta\gamma} X_\gamma \right] \\ &= \{f_0, f_0^*\}_{\mathbb{R}} \frac{\partial}{\partial t} + \sum_{\gamma=1}^r \left[\left(f_0 \frac{df_\gamma^*}{dt} - f_0^* \frac{df_\gamma}{dt} \right) + \sum_{\alpha, \beta=1}^r f_\alpha f_\beta^* c_{\alpha\beta\gamma} \right] X_\gamma. \end{aligned}$$

Since S_X^V is a Lie algebra and $Y, Y^* \in S_X^V$, then $[Y, Y^*] \in S_X^V$. As additionally $\{f_0, f_0^*\}_{\mathbb{R}} \in \mathfrak{W}$, and then $[Y, Y^*] \in S_{X, \mathfrak{W}}^V$. Hence, the Lie bracket of elements of $S_{X, \mathfrak{W}}^V$ belongs to $S_{X, \mathfrak{W}}^V$, which becomes a Lie algebra. \square

Corollary 4.4. Given a Lie system X on N related to a Vessiot–Guldberg Lie algebra V , the elements of $S_{X, \mathfrak{W}}^V$ with

1. $\mathfrak{W} = \{f_0 \in C^\infty(\mathbb{R}) \mid df_0/dt = 0\}$,
2. $\mathfrak{W} = \{f_0 \in C^\infty(\mathbb{R}) \mid f_0 = 0\}$,

are finite-dimensional Lie algebras of vector fields. In the second case, $S_{X, \mathfrak{W}}^V$ is isomorphic to V .

Proof. In both cases, \mathfrak{W} is nonempty. In the first case, the functions f_0 with $df_0/dt = 0$ are constant. These functions form an Abelian Lie algebra with respect to the Lie bracket $\{\cdot, \cdot\}_{\mathbb{R}}$. In view of Proposition 4.3, the space $S_{X, \mathfrak{W}}^V$ is a Lie algebra.

In the second case, the function zero is also a zero-dimensional Lie algebra relative to $\{\cdot, \cdot\}_{\mathbb{R}}$. Since $f_0 = 0$ and using Proposition 4.3, we obtain that $S_{X, \mathfrak{W}}^V$ can be understood as a Lie algebra of t -dependent vector fields taking values in V . To prove that $S_{X, \mathfrak{W}}^V \simeq V$, let us consider the morphism which maps each t -dependent vector field with its value at $t = 0$, namely

$$\begin{aligned} \phi : S_{X, \mathfrak{W}}^V &\longrightarrow V \\ Z &\longmapsto Z_0. \end{aligned}$$

Let X_1, \dots, X_r be a basis for V . From Lemma (3.1), we have that $(f_1(t), \dots, f_r(t))$ is a particular solution of the system

$$\frac{df_\alpha}{dt} = \sum_{\delta, \beta=1}^r b_\beta(t) f_\delta c_{\delta\beta\alpha}, \quad \alpha = 1, \dots, r.$$

For each initial condition $f_\alpha(0) = c_\alpha \in \mathbb{R}$, with $\alpha = 1, \dots, r$, i.e., by fixing Z_0 , there exists a unique solution of the above system. Hence, there exists a unique t -dependent vector field Z of $S_{X, \mathfrak{W}}^V$ with $Z_0 = \sum_{\alpha=0}^r c_\alpha X_\alpha$. Thus, ϕ is a bijection. Using that for two vector fields $Z_1, Z_2 \in S_{X, \mathfrak{W}}^V$ we have $[Z_1, Z_2] \in S_{X, \mathfrak{W}}^V$ and $[Z_1, Z_2]_t = [(Z_1)_t, (Z_2)_t]$, we see that ϕ is a Lie algebra morphism and $S_{X, \mathfrak{W}}^V \simeq V$. \square

5. Applications to systems of ODEs and HODES

Let us work out the symmetry systems and related Lie symmetries for some Lie systems of interest. In particular, we will illustrate that Proposition 3.5 enables us to determine simultaneously Lie symmetries for different Lie systems with isomorphic Vessiot–Guldberg Lie algebras.

5.1. Lie symmetries for $sl(2, \mathbb{R})$ -Lie systems

Let us obtain the symmetry systems and related Lie symmetries for $sl(2, \mathbb{R})$ -Lie systems. This shall be used in following sections to obtain simultaneously Lie symmetries of isomorphic $sl(2, \mathbb{R})$ -Lie systems appearing in the physics and/or mathematical literature.

Let us choose a basis of vector fields $\{X_1, X_2, X_3\}$ of $V \simeq sl(2, \mathbb{R})$ with commutation relations

$$[X_1, X_2] = X_1, \quad [X_1, X_3] = 2X_2, \quad [X_2, X_3] = X_3. \tag{5.1}$$

Every Lie system with Vessiot–Guldberg Lie algebra V can be brought into the form

$$X = b_1(t)X_1 + b_2(t)X_2 + b_3(t)X_3 \tag{5.2}$$

for certain t -dependent functions b_1, b_2 and b_3 . The Lie symmetries of S_X^V take the form

$$Y = f_0(t) \frac{\partial}{\partial t} + f_1(t)X_1 + f_2(t)X_2 + f_3(t)X_3,$$

where f_0, f_1, f_2, f_3 are some t -dependent functions to be determined. In view of (3.3) and the commutation relations (5.1), the symmetry system for X relative to V reads

$$\begin{cases} \frac{df_0}{dt} = b_0(t), \\ \frac{df_1}{dt} = f_0 \frac{db_1}{dt}(t) + f_1 b_2(t) - f_2 b_1(t) + b_0(t) b_1(t), \\ \frac{df_2}{dt} = f_0 \frac{db_2}{dt}(t) + 2f_1 b_3(t) - 2f_3 b_1(t) + b_0(t) b_2(t), \\ \frac{df_3}{dt} = f_0 \frac{db_3}{dt}(t) + f_2 b_3(t) - f_3 b_2(t) + b_0(t) b_3(t). \end{cases} \tag{5.3}$$

As stated in Theorem 3.3, this is a Lie system. Indeed, system (5.3) is related to the t -dependent vector field

$$\Gamma_X^{sl(2, \mathbb{R})} = \frac{db_1(t)}{dt} W_1 + \frac{db_2(t)}{dt} W_2 + \frac{db_3(t)}{dt} W_3 + b_0(t) Z_0 + b_0(t) b_1(t) Z_1 + b_0(t) b_2(t) Z_2 + b_0(t) b_3(t) Z_3 + b_1(t) Y_1 + b_2(t) Y_2 + b_3(t) Y_3,$$

where

$$Z_\alpha = \frac{\partial}{\partial f_\alpha}, \quad \alpha = 0, 1, 2, 3, \quad W_\beta = f_0 \frac{\partial}{\partial f_\beta}, \quad \beta = 1, 2, 3 \tag{5.4}$$

and

$$Y_1 = -f_2 \frac{\partial}{\partial f_1} - 2f_3 \frac{\partial}{\partial f_2}, \quad Y_2 = f_1 \frac{\partial}{\partial f_1} - f_3 \frac{\partial}{\partial f_3}, \quad Y_3 = 2f_1 \frac{\partial}{\partial f_2} + f_2 \frac{\partial}{\partial f_3}. \tag{5.5}$$

These vector fields hold

$$[Y_1, Y_2] = Y_1, \quad [Y_1, Y_3] = 2Y_2, \quad [Y_2, Y_3] = Y_3. \tag{5.6}$$

Since $Z(V) = \{0\}$, then $V_L = \langle Y_1, Y_2, Y_3 \rangle$ is a Lie algebra isomorphic to $V/Z(V) \simeq sl(2, \mathbb{R})$ as stated in Theorem 3.3. The rest of commutation relations read

$$\begin{aligned} [Y_1, Z_0] &= 0, & [Y_1, Z_1] &= 0, & [Y_1, Z_2] &= Z_1, & [Y_1, Z_3] &= 2Z_2, \\ [Y_2, Z_0] &= 0, & [Y_2, Z_1] &= -Z_1, & [Y_2, Z_2] &= 0, & [Y_2, Z_3] &= Z_3, \\ [Y_3, Z_0] &= 0, & [Y_3, Z_1] &= -2Z_2, & [Y_3, Z_2] &= -Z_3, & [Y_3, Z_3] &= 0. \end{aligned}$$

Moreover,

$$\begin{aligned} [Y_1, W_1] &= 0, & [Y_1, W_2] &= W_1, & [Y_1, W_3] &= 2W_2 \\ [Y_2, W_1] &= -W_1, & [Y_2, W_2] &= 0, & [Y_2, W_3] &= W_3, \\ [Y_3, W_1] &= -2W_2, & [Y_3, W_2] &= -W_3, & [Y_3, W_3] &= 0 \end{aligned}$$

and

$$\begin{aligned} [Z_0, W_j] &= Z_j, & [Z_i, W_j] &= 0, & [W_i, W_j] &= 0, & i, j &= 1, 2, 3, \\ [Z_\alpha, Z_\beta] &= 0, & \alpha, \beta &= 0, \dots, 3. \end{aligned} \tag{5.7}$$

Hence, $A_1 = \langle Z_0, Z_1, Z_2, Z_3 \rangle$ is an ideal of $A_1 + A_2$. And $A_1 + A_2$ is an ideal of $A_1 + A_2 + V_L$, with $A_2 = \langle W_1, W_2, W_3 \rangle$.

5.1.1. A first-order Riccati equation

The previous results can be applied in studying the first-order Riccati equation [24,31]

$$\frac{dx}{dt} = \eta(t) + x^2, \quad (5.8)$$

where η is an arbitrary t -dependent function.

It is well known that the Riccati equation is a Lie system with a Vessiot–Guldberg Lie algebra isomorphic to $V^{\text{Ric}} \simeq \mathfrak{sl}(2, \mathbb{R})$ [13]. Indeed, Eq. (5.8) has the associated t -dependent vector field $X^{\text{Ric}} = X_3^{\text{Ric}} + \eta(t)X_1^{\text{Ric}}$, where

$$X_1^{\text{Ric}} = \frac{\partial}{\partial x}, \quad X_2^{\text{Ric}} = x \frac{\partial}{\partial x}, \quad X_3^{\text{Ric}} = x^2 \frac{\partial}{\partial x} \quad (5.9)$$

satisfy the same commutation relations as X_1, X_2, X_3 in (5.1). In view of this, (5.8) is related to a t -dependent vector field taking values in a finite-dimensional Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Then, (5.8) is an $\mathfrak{sl}(2, \mathbb{R})$ -Lie system. Moreover, we can consider X^{Ric} as a particular case of system (5.2). By applying the results of the previous sections to generic $\mathfrak{sl}(2, \mathbb{R})$ -Lie systems, we find that the symmetry system for X^{Ric} is of the form (5.3) with $b_1 = \eta$, $b_2 = 0$ and $b_3 = 1$, namely

$$\begin{cases} \frac{df_0}{dt} = b_0(t), \\ \frac{df_1}{dt} = f_0 \frac{d\eta}{dt}(t) - \eta(t)f_2 + b_0(t)\eta(t), \\ \frac{df_2}{dt} = 2f_1 - 2\eta(t)f_3, \\ \frac{df_3}{dt} = f_2 + b_0(t). \end{cases} \quad (5.10)$$

We can recover and generalize the results given in [21] by means of our approach. From their expressions in (5.10), we can differentiate df_3/dt twice and df_2/dt once. By substituting d^2f_2/dt^2 into d^3f_3/dt^3 and using the remaining equations in (5.10), we obtain that

$$\frac{d^3f_3}{dt^3} = \frac{d^3f_0}{dt^3} + 2\frac{df_1}{dt} - 2\frac{d\eta}{dt}f_3 - 2\eta(t)\frac{df_3}{dt}.$$

By substituting the value of df_1/dt from (5.10) and using that $f_2 = df_3/dt - b_0(t)$, we obtain the following equation for f_3 in terms of the coefficients of (5.8) and f_0 :

$$\frac{d^3f_3}{dt^3} = \frac{d^3f_0}{dt^3} + 4b_0\eta(t) + 2\frac{d\eta}{dt}f_0 - 2\frac{d\eta}{dt}f_3 - 4\eta(t)\frac{df_3}{dt}. \quad (5.11)$$

From this, we can retrieve the following corollary given in [21].

Corollary 5.1. *The Riccati equation (5.8) admits the Lie symmetry*

$$Y = f_0 \frac{\partial}{\partial t} - \frac{1}{2} \frac{d^2f_0}{dt^2} \frac{\partial}{\partial x} - \frac{df_0}{dt} x \frac{\partial}{\partial x},$$

where

$$\frac{d^3f_0}{dt^3} + 4b_0(t)\eta(t) + 2\frac{d\eta}{dt}f_0 = 0. \quad (5.12)$$

Proof. Since we are looking for Lie symmetries with $f_3 = 0$, Eq. (5.11) reduces to (5.12). Moreover, by substituting $f_3 = 0$ in (5.10), we obtain that

$$\frac{df_3}{dt} = b_0(t) + f_2 = 0, \quad \frac{df_2}{dt} = 2f_1,$$

which yields $f_2 = -df_0/dt$ and $2f_1 = -d^2f_0/dt^2$. Hence, the corollary follows. \square

Going back to general symmetries of (5.8), we can obtain some of its Lie symmetries by solving (5.10) for certain values of $\eta(t)$ summarized in Table 1.

5.1.2. Cayley–Klein Riccati equation

Let us generalize the results of the previous section to a more general family of Riccati equations over different \mathbb{R} -linear algebras of numbers, namely real vector spaces along with a certain bilinear product. In what follows, we summarize roughly some results given in [36] (see from page 265).

Consider the real plane with coordinates $\{x, y\}$ and an “additional” unit ι such that

$$\iota^2 \in \{-1, +1, 0\}. \quad (5.13)$$

Table 1

Lie symmetries for Riccati Eqs. (5.8) for values of the function $\eta(t)$. We assume $f_0 = k \in \mathbb{R}$. Function f_3 can be derived from (5.12). Function f_2 can be obtained from f_3 , as $f_2 = df_3/dt$. Function f_1 reads $f_1 = k\eta(t) - \int \eta(t)f_3 dt$. In the following Airy_A and Airy_B denote the Airy and Bairy functions and J, Y are the Bessel functions of first and second kind.

η	$f_3(t)$
$\frac{k}{at+b}$	$k + c_1(at+b)^2(1, 2\sqrt{(at+b)/a^2}) + c_2(at+b)Y^2(1, 2\sqrt{at+b/a^2}) + c_3(at+b)J(1, 2\sqrt{t+b/a})Y(1, 2\sqrt{t+b/a})$
$\frac{k}{(at+b)^2}$	$-\frac{k at}{b} + c_1\left(\frac{at+b}{a}\right)^{\frac{a+\sqrt{a^2-4k}}{a}} + c_2\left(\frac{at+b}{a}\right)^{-\frac{a-\sqrt{a^2-4k}}{a}} + c_3(at+b)$
$at+b$	$k + c_1 \text{Airy}_A\left(-\frac{at+b}{a^{2/3}}\right)^2 + c_2 \text{Airy}_B\left(-\frac{at+b}{a^{2/3}}\right)^2 + c_3 \text{Airy}_A\left(-\frac{at+b}{a^{2/3}}\right) \text{Airy}_B\left(-\frac{at+b}{a^{2/3}}\right)$

Next, we define the real vector space of elements z of the form

$$z := x + \iota y, \tag{5.14}$$

with the standard addition and multiplication: $(x_1 + \iota y_1) + (x_2 + \iota y_2) = x_1 + x_2 + \iota(y_1 + y_2)$ and $\lambda(x_1 + \iota y_1) := \lambda x_1 + \iota \lambda y_1$ for arbitrary $x_1, x_2, y_1, y_2, \lambda \in \mathbb{R}$. We define $\bar{z} := x - \iota y$ and a bilinear multiplication in the usual way: $(x_1 + \iota y_1)(x_2 + \iota y_2) = x_1 x_2 + \iota^2 y_1 y_2 + \iota(x_1 y_2 + y_1 x_2)$. Hence,

$$|z|^2 = z\bar{z} = x^2 - \iota^2 y^2, \quad z^2 = x^2 + \iota^2 y^2 + 2\iota xy. \tag{5.15}$$

In this way, we obtain three different algebra structures on \mathbb{R}^2 over \mathbb{R} , namely:

- Case $\iota^2 = -1$. We say that ι is an *elliptical* additional unit. Here we are dealing with the usual complex numbers: $\iota \equiv i$ and $z \in \mathbb{C}$. Hence,

$$|z|^2 = z\bar{z} = x^2 + y^2, \quad z^2 = x^2 - y^2 + 2\iota xy, \quad z \in \mathbb{C}. \tag{5.16}$$

- Case $\iota^2 = +1$. We call ι a *hyperbolic* additional unit and we are dealing with the so-called *split complex numbers* $z \in \mathbb{C}'$. The additional unit is usually known as *double* or *Clifford* unit $\iota \equiv e$ [36]. Thus,

$$|z|^2 = z\bar{z} = x^2 - y^2, \quad z^2 = x^2 + y^2 + 2\epsilon xy, \quad z \in \mathbb{C}'. \tag{5.17}$$

- Case $\iota^2 = 0$. This last possibility corresponds to a *parabolic* additional unit, $\iota \equiv \epsilon$, and z is known as a *dual* or *Study* number $z \in \mathbb{D}$ [36] which can be regarded as a *contracted* case since

$$|z|^2 = z\bar{z} = x^2, \quad z^2 = x^2 + 2\epsilon xy, \quad z \in \mathbb{D}. \tag{5.18}$$

We define the *Cayley–Klein Riccati equation* to be the generalization of the standard Riccati equation to $z \in \{\mathbb{C}, \mathbb{C}', \mathbb{D}\}$:

$$\frac{dz}{dt} = b_1(t) + b_2(t)z + b_3(t)z^2, \quad z = x + \iota y. \tag{5.19}$$

For real t -dependent coefficients b_1, b_2 and b_3 , this equation gives rise to a system X^{CK} of two real differential equations

$$\frac{dx}{dt} = b_1(t) + b_2(t)x + b_3(t)(x^2 + \iota^2 y^2), \quad \frac{dy}{dt} = b_2(t)y + 2b_3(t)xy. \tag{5.20}$$

The system (5.20) can be described by means of the t -dependent vector field

$$X^{\text{CK}} = b_1(t)X_1^{\text{CK}} + b_2(t)X_2^{\text{CK}} + b_3(t)X_3^{\text{CK}},$$

where

$$X_1^{\text{CK}} = \frac{\partial}{\partial x}, \quad X_2^{\text{CK}} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \quad X_3^{\text{CK}} = (x^2 + \iota^2 y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y}$$

satisfy the same commutation relations as X_1, X_2, X_3 in (5.1). Since X_1, X_2, X_3 span a Lie algebra $V \simeq \mathfrak{sl}(2, \mathbb{R})$, the vector fields $X_1^{\text{CK}}, X_2^{\text{CK}}$ and X_3^{CK} span a real Lie algebra, V^{CK} , which is isomorphic to the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ independently of the square of ι . Then, (5.20) is an $\mathfrak{sl}(2, \mathbb{R})$ -Lie system.

Moreover, we can define a the Lie algebra isomorphism $\phi: V^{\text{CK}} \rightarrow V$ mapping

$$\phi(X_\alpha^{\text{CK}}) = X_\alpha, \quad \alpha = 1, 2, 3,$$

where X_α , with $\alpha = 1, 2, 3$, are the generic vector fields in (5.1). Hence, we have that $\phi(X_t^{\text{CK}}) = X_t$ for every $t \in \mathbb{R}$ with X being the generic $\mathfrak{sl}(2, \mathbb{R})$ -Lie system (5.2). Then, (5.20) is isomorphic to (5.2) and, in view of Proposition 3.5, the symmetry system of X^{CK} becomes (5.3).

5.1.3. Quaternionic Riccati equation with t -dependent real coefficients

Quaternions are the elements of the real vector space $\mathbb{H} \simeq \mathbb{R}^4$ of elements of the form $q = q_0 + q_1i + q_2j + q_3k$, where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and with the standard sum of elements and multiplication by real numbers. We can define a multiplication of quaternions (see [18] for details) by assuming that real numbers commute with all quaternions and that the following operations are fulfilled

$$i^2 = -1, \quad j^2 = -1, \quad k^2 = -1, \\ i \cdot j = -j \cdot i = k, \quad k \cdot i = -i \cdot k = j, \quad j \cdot k = -k \cdot j = i.$$

The quaternionic Riccati equation takes the form (see [34] for details)

$$\frac{dq}{dt} = b_1(t) + a_1(t)q + qa_2(t) + qb_3(t)q, \tag{5.21}$$

where q and the t -dependent functions $a_1, a_2, b_1, b_3 : \mathbb{R} \rightarrow \mathbb{H}$ take values in \mathbb{H} [18]. The existence of periodic solutions for particular cases of (5.21) has been studied in [6,34] and, for real t -dependent coefficients, in [34]. In this work, we focus on the latter case.

Writing q in coordinates, we obtain that (5.21) reads

$$\begin{cases} \frac{dq_0}{dt} = b_1(t) + b_2(t)q_0 + b_3(t)(q_0^2 - q_1^2 - q_2^2 - q_3^2), \\ \frac{dq_1}{dt} = b_2(t)q_1 + 2b_3(t)q_0q_1, \\ \frac{dq_2}{dt} = b_2(t)q_2 + 2b_3(t)q_0q_2, \\ \frac{dq_3}{dt} = b_2(t)q_3 + 2b_3(t)q_0q_3, \end{cases} \tag{5.22}$$

where $b_2(t) \equiv a_1(t) + a_2(t)$. This system is related to the t -dependent vector field

$$X^{\mathbb{H}} = b_1(t)X_1^{\mathbb{H}} + b_2(t)X_2^{\mathbb{H}} + b_3(t)X_3^{\mathbb{H}}, \tag{5.23}$$

with

$$X_1^{\mathbb{H}} = \frac{\partial}{\partial q_0}, \quad X_2^{\mathbb{H}} = q_0 \frac{\partial}{\partial q_0} + q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2} + q_3 \frac{\partial}{\partial q_3}, \\ X_3^{\mathbb{H}} = (q_0^2 - q_1^2 - q_2^2 - q_3^2) \frac{\partial}{\partial q_0} + 2q_0 \left(q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2} + q_3 \frac{\partial}{\partial q_3} \right),$$

which satisfy the same commutation relations as X_1, X_2 and X_3 in (5.1). So, $X_1^{\mathbb{H}}, X_2^{\mathbb{H}}, X_3^{\mathbb{H}}$ span a real Lie algebra $V^{\mathbb{H}}$ isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Hence, the quaternionic Riccati Eq. (5.21) with t -dependent real coefficients is an $\mathfrak{sl}(2, \mathbb{R})$ -Lie system isomorphic to (5.2) with respect to the Lie algebra isomorphism $\phi : V^{\mathbb{H}} \rightarrow V$ being given by

$$\phi(X_\alpha^{\mathbb{H}}) = X_\alpha, \quad \alpha = 1, 2, 3.$$

Moreover, we have that $\phi(X_t^{\mathbb{H}}) = X_t$ for every $t \in \mathbb{R}$. Hence, the symmetry system for (5.21) is (5.3). If we assume for instance $f_0 = k \in \mathbb{R}$, and $b_1(t) = \eta(t)$, $b_2(t) = 0$ and $b_3(t) = 1$, we obtain that (5.21) is isomorphic to the Lie system (5.10). Hence, for certain values of $\eta(t)$ given in Table 1, we can derive several Lie symmetries for quaternionic Riccati equations.

5.1.4. The generalized Darboux–Brioschi–Halphen system

We now show that our theory in particular, and the whole theory of Lie systems in general, can be used to study autonomous systems of first-order ODEs. We consider the generalized Darboux–Brioschi–Halphen system (DBH system) [19]:

$$\begin{cases} \frac{dw_1}{dt} = w_3w_2 - w_1w_3 - w_1w_2 + \tau^2, \\ \frac{dw_2}{dt} = w_1w_3 - w_2w_1 - w_2w_3 + \tau^2, \\ \frac{dw_3}{dt} = w_2w_1 - w_3w_2 - w_3w_1 + \tau^2, \end{cases} \tag{5.24}$$

where

$$\tau^2 \equiv \alpha_1^2(\omega_1 - \omega_2)(\omega_3 - \omega_1) + \alpha_2^2(\omega_2 - \omega_3)(\omega_1 - \omega_2) + \alpha_3^2(\omega_3 - \omega_1)(\omega_2 - \omega_3)$$

and $\alpha_1, \alpha_2, \alpha_3$ are real constants.

When $\tau = 0$, system (5.24) retrieves the classical DBH system solved by Halphen [14,19,23] which appears in the study of triply orthogonal surfaces and the vacuum Einstein equations for hyper-Kähler Bianchi-IX metrics. For $\tau \neq 0$, the generalized

DBH system can be considered as a reduction of the self-dual Yang–Mills equations corresponding to an infinite-dimensional gauge group of diffeomorphisms of a three-dimensional sphere [14].

It can be proved that (5.24) is an $\mathfrak{sl}(2, \mathbb{R})$ -Lie system. Indeed, it is the associated system to the t -dependent vector field

$$X_t^{\text{DBH}} = (w_3w_2 - w_1(w_3 + w_2) + \tau^2) \frac{\partial}{\partial w_1} + (w_1w_3 - w_2(w_1 + w_3) + \tau^2) \frac{\partial}{\partial w_2} + (w_2w_1 - w_3(w_2 + w_1) + \tau^2) \frac{\partial}{\partial w_3} = -X_3^{\text{DBH}}.$$

This vector field span a Lie algebra V^{DBH} of vector fields along with

$$X_1^{\text{DBH}} = \frac{\partial}{\partial w_1} + \frac{\partial}{\partial w_2} + \frac{\partial}{\partial w_3}, \quad X_2^{\text{DBH}} = w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2} + w_3 \frac{\partial}{\partial w_3}$$

satisfying the commutation relations (5.1). In consequence, $X_1^{\text{DBH}}, X_2^{\text{DBH}}$ and X_3^{DBH} span a three-dimensional Lie algebra of vector fields V^{DBH} isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ and then X^{DBH} is an $\mathfrak{sl}(2, \mathbb{R})$ -Lie system. Since X_1, X_2, X_3 admit the same structure constants as (5.1), the symmetry system for X^{DBH} becomes (5.3) with $b_1(t) = b_2(t) = 0$ and $b_3(t) = -1$, namely

$$\frac{df_0}{dt} = b_0(t), \quad \frac{df_1}{dt} = 0, \quad \frac{df_2}{dt} = -2f_1, \quad \frac{df_3}{dt} = -f_2 - b_0(t).$$

Hence, for $b_0(t) = 0$, we obtain

$$Y = t_0 \frac{\partial}{\partial t} + \lambda_1 X_1 - (2\lambda_1 t - \lambda_2) X_2 + (\lambda_1 t^2 - \lambda_2 t + \lambda_3) X_3, \quad \lambda_1, \lambda_2, \lambda_3, t_0 \in \mathbb{R}.$$

Evidently, these vector fields span a Lie algebra of Lie symmetries isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ for $t_0 = 0$. For $b_0(t) = c_0$, we obtain

$$Y = (c_0 t + t_0) \frac{\partial}{\partial t} + \lambda_1 X_1 - (2\lambda_1 t - \lambda_2) X_2 + (\lambda_1 t^2 - (\lambda_2 + c_0)t + \lambda_3) X_3, \quad \lambda_1, \lambda_2, \lambda_3, t_0, c_0 \in \mathbb{R}.$$

Finally, for $b_0(t) = c_0 t$, we get

$$Y = \left(t_0 + \frac{c_0 t^2}{2} \right) \frac{\partial}{\partial t} + \lambda_1 X_1 - (2\lambda_1 t - \lambda_2) X_2 + \left[\lambda_1 t^2 - \left(\lambda_2 + \frac{c_0 t}{2} \right) t + \lambda_3 \right] X_3,$$

with $\lambda_1, \lambda_2, \lambda_3, t_0, c_0 \in \mathbb{R}$. Since $\mathfrak{W} = \langle 1, t, t^2 \rangle$ is a Lie algebra with respect to the Lie bracket, $\{\cdot, \cdot\}_{\mathbb{R}}$, we obtain in view of Proposition 4.3 that

$$S_{X, \mathfrak{W}}^V = \{Y \in S_X^V \mid f_0 \in \langle 1, t, t^2 \rangle\}$$

is a Lie algebra of Lie symmetries. By choosing appropriately the constant coefficients of the above vector fields and setting $\tau = 0$, we recover the Lie algebra of symmetries isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ for classical DBH systems given by Nucci in [28, p. 295].

5.2. Applications to systems of HODEs

The study of systems of HODEs through Lie systems implies the addition of extra variables in order to express a higher-order system as a first-order one. The introduction of these extra variables frequently results in the obtainment of non-local Lie symmetries, as we shall exemplify in forthcoming examples. Certain non-local Lie symmetries can be identified with the prolongations of certain vector fields.

Lemma 5.2. *Given a Lie algebra of Lie point symmetries V , its prolongations form a Lie algebra \widehat{V} isomorphic to the former.*

Proof. Consider the map

$$\begin{aligned} \Phi : \Gamma(T(\mathbb{R} \times N)) &\rightarrow \Gamma(T[\mathbb{R} \times T^p N]) \\ X &\mapsto \widehat{X}, \end{aligned}$$

mapping sections of $T(\mathbb{R} \times N)$, i.e., vector fields on $\mathbb{R} \times N$, onto sections of $T(\mathbb{R} \times T^p N)$, i.e., vector fields on the manifold $\mathbb{R} \times T^p N$, where $T^p N$ is the so-called p -order tangent space [27]. Roughly speaking, $T^p N$ is the space of equivalence classes of curves in N with the same Taylor expansion up to order p . It admits a differentiable structure induced by the variables x_1, \dots, x_n of the coordinate systems of N and the induced variables $x_i^{(k)}$ describing the derivatives in terms of t up to order p of the coordinates of a curve within N . Given the so-called contact one-forms $\theta_i^k = dx_i^{(k)} - x_i^{(k+1)} dt$ with $i = 1, \dots, n$ and $k = 0, \dots, p - 1$ on $\mathbb{R} \times T^p N$, we say that \widehat{X} is the prolongation of $X \in \Gamma(T(\mathbb{R} \times N))$ to $\mathbb{R} \times T^p N$ if and only if every $\mathcal{L}_{\widehat{X}} \theta_i^k$ belongs to the contact ideal spanned by all the contact forms and $J_{\pi_*}^p \widehat{X} = X$, with $J_{\pi_*}^p$ being the tangent map to the projection $J_{\pi}^p : (t, x_i, x_i^{(k)}) \in \mathbb{R} \times T^p N \mapsto (t, x_i) \in \mathbb{R} \times N$. This implies that Φ is \mathbb{R} -linear and injective. Additionally,

$$\mathcal{L}_{[\widehat{X}_1, \widehat{X}_2]} \theta_i^k = (\mathcal{L}_{\widehat{X}_1} \mathcal{L}_{\widehat{X}_2} - \mathcal{L}_{\widehat{X}_2} \mathcal{L}_{\widehat{X}_1}) \theta_i^k$$

belongs to the ideal of the contact forms because $\mathcal{L}_{\widehat{X}_1}$ and $\mathcal{L}_{\widehat{X}_2}$ do so. Moreover,

$$J_{\pi_*}^p[\widehat{X}_1, \widehat{X}_2] = [J_{\pi_*}^p\widehat{X}_1, J_{\pi_*}^p\widehat{X}_2] = [X_1, X_2].$$

Hence, $[\widehat{X}_1, \widehat{X}_2]$ must be the prolongation of $[X_1, X_2]$, i.e., $[\widehat{X}_1, \widehat{X}_2] = \widehat{[X_1, X_2]}$. In this way, Φ is a Lie algebra morphism. Obviously, given a Lie algebra of Lie point symmetries V , its prolongations form a Lie algebra $\widehat{V} = \Phi(V)$ isomorphic to V . \square

Definition 5.3. We call \widehat{V} the Lie algebra whose elements are the prolongations of the elements in V .

5.2.1. Second-order Kummer–Schwarz equation

As an example of a second-order Lie system possessing a Vessiot–Gulberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, we present the second-order Kummer–Schwarz equation

$$\frac{d^2x}{dt^2} = \frac{3}{2x} \left(\frac{dx}{dt} \right)^2 - 2c_0x^3 + 2\eta(t)x, \tag{5.25}$$

where c_0 is a real constant and $\eta(t)$ is an arbitrary t -dependent function. The interest of this differential equation is due to its appearance in the study of other remarkable (systems of) ODEs, e.g., Riccati and Ermakov systems, with physical and mathematical applications [8,17]. It is well known that (5.25) is a Lie system when it is rewritten as a first-order system by introducing an additional variable $v \equiv dx/dt$, i.e.,

$$\begin{cases} \frac{dx}{dt} = v, \\ \frac{dv}{dt} = \frac{3}{2} \frac{v^2}{x} - 2c_0x^3 + 2\eta(t)x. \end{cases} \tag{5.26}$$

This system is associated with the t -dependent vector field

$$M^{KS} = M_3^{KS} + \eta(t)M_1^{KS},$$

where the vector fields

$$M_1^{KS} = 2x \frac{\partial}{\partial v}, \quad M_2^{KS} = x \frac{\partial}{\partial x} + 2v \frac{\partial}{\partial v}, \quad M_3^{KS} = v \frac{\partial}{\partial x} + \left(\frac{3}{2} \frac{v^2}{x} - 2c_0x^3 \right) \frac{\partial}{\partial v}$$

satisfy the same commutation relations as the vector fields X_1, X_2, X_3 in (5.1) and they therefore span a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Since the basis $M_1^{KS}, M_2^{KS}, M_3^{KS}$ have the same structure constants as the $\mathfrak{sl}(2, \mathbb{R})$ -Lie systems analyzed in Section 5.1, we can define, for instance, a Lie algebra morphism ϕ mapping $M_1^{KS}, M_2^{KS}, M_3^{KS}$ to the basis $X_1^{Ric}, X_2^{Ric}, X_3^{Ric}$ for Riccati equations (5.8). In such a case, M^{KS} maps to X^{Ric} and in view of Proposition 3.5, the symmetry system for M^{KS} is of the form of the symmetry system (5.10). As a consequence, the particular solutions f_0, f_1, f_2 and f_3 for Riccati equations, detailed in Table 1, are valid for (5.26) as well.

5.3. Lie symmetries for $\text{Aff}(\mathbb{R})$ -Lie systems

In this section, we aim to obtain elements of S_X^V for $\text{Aff}(\mathbb{R})$ -Lie systems. We choose a basis of vector fields $\{X_1, X_2\}$ of $V \simeq \text{Aff}(\mathbb{R})$ with $[X_1, X_2] = X_1$, and express the most general Lie system with Vessiot–Guldberg Lie algebra V as $X = a(t)X_1 + b(t)X_2$. Let us now look for its Lie symmetries of the form

$$Y = f_0(t) \frac{\partial}{\partial t} + f_1(t)X_1 + f_2(t)X_2. \tag{5.27}$$

The symmetry condition gives rise to a symmetry system

$$\begin{cases} \frac{df_0}{dt} = b_0(t), \\ \frac{df_1}{dt} = f_0 \frac{da}{dt}(t) + a(t)b_0(t) + b(t)f_1 - a(t)f_2, \\ \frac{df_2}{dt} = f_0 \frac{db}{dt}(t) + b(t)b_0(t), \end{cases} \tag{5.28}$$

associated with the t -dependent vector field

$$\Gamma_X^{\text{Aff}(\mathbb{R})} = b_0(t)Z_0 + \frac{da}{dt}W_1 + \frac{db}{dt}W_2 + b_0(t)a(t)Z_1 + b_0(t)b(t)Z_2 + a(t)Y_1 + b(t)Y_2$$

where

$$Y_1 = -f_2 \frac{\partial}{\partial f_1}, \quad Y_2 = f_1 \frac{\partial}{\partial f_1}, \quad W_1 = f_0 \frac{\partial}{\partial f_1}, \quad W_2 = f_0 \frac{\partial}{\partial f_2}, \quad Z_\alpha = \frac{\partial}{\partial f_\alpha}, \quad \alpha = 0, 1, 2.$$

Since, $[Y_1, Y_2] = Y_1$, then $V_L = \langle Y_1, Y_2 \rangle$ gives rise to a Lie algebra isomorphic to $\text{Aff}(\mathbb{R})$. Moreover,

$$\begin{aligned} [Z_0, W_1] &= Z_1, & [Z_0, W_2] &= Z_2, & [Z_i, W_j] &= 0, & i, j &= 1, 2, & [Z_\alpha, Z_\beta] &= 0, & \alpha, \beta &= 0, 1, 2, \\ [Y_1, W_1] &= 0, & [Y_1, W_2] &= W_1, & [Y_1, Z_0] &= 0, & [Y_1, Z_1] &= 0, & [Y_1, Z_2] &= Z_1, \\ [Y_2, W_1] &= -W_1, & [Y_2, W_2] &= 0, & [Y_2, Z_0] &= 0, & [Y_2, Z_1] &= -Z_1, & [Y_2, Z_2] &= 0. \end{aligned} \tag{5.29}$$

In this way, $A_1 = \langle Z_0, Z_1, Z_2 \rangle$ is an ideal of $A_1 + A_2$, with $A_2 = \langle W_1, W_2 \rangle$ and $A_1 \oplus_S A_2$ is an ideal of $A_1 + A_2 + V_L$. Hence, system (5.28) possesses a Vessiot–Guldberg Lie algebra

$$(A_1 \oplus_S A_2) \oplus_S V_L \simeq (\mathbb{R}^3 \oplus_S \mathbb{R}^2) \oplus_S \text{Aff}(\mathbb{R}). \tag{5.30}$$

We can solve (5.28) when $b_0(t) = 0$. In this case, $f_0 = k \in \mathbb{R}$ and (5.28) reduces to a trivial equation for f_2 and a linear one for f_1 . The general solution reads:

$$\begin{aligned} f_1(t) &= \left[\int_0^t \left[k \frac{da}{dt'}(t') - a(t')(kb(t') + c_1) \right] e^{-\int_0^{t'} b(t'') dt''} dt' + c_2 \right] e^{\int_0^t b(t') dt'}, \\ f_2(t) &= kb(t) + c_1, \end{aligned} \tag{5.31}$$

where c_1 and c_2 are constants of integration.

In order to illustrate $\text{Aff}(\mathbb{R})$ -Lie systems through a physical example, we consider the Buchdahl equation [5,15,16]

$$\frac{d^2x}{dt^2} = f(x) \left(\frac{dx}{dt} \right)^2 + a_2(t) \frac{dx}{dt},$$

where f is any function depending on x and a_2 is any t -dependent function. The interest of this system is due to its appearance in General Relativity [16]. We add a new variable $v \equiv dx/dt$ to obtain

$$\begin{cases} \frac{dx}{dt} = v, \\ \frac{dv}{dt} = f(x)v^2 + a_2(t)v, \end{cases} \tag{5.32}$$

the hereafter called *Buchdahl system*. It is known that (5.32) is a Lie system [2]. Indeed, (5.32) describes the integral curves of the t -dependent vector field

$$X^{BD} = v \frac{\partial}{\partial x} + (f(x)v^2 + a_2(t)v) \frac{\partial}{\partial v} = X_1 - a_2(t)X_2,$$

where

$$X_1^{BD} = v \frac{\partial}{\partial x} + f(x)v^2 \frac{\partial}{\partial v}, \quad X_2^{BD} = -v \frac{\partial}{\partial v},$$

satisfy $[X_1, X_2] = X_1$. That is, X is an $\text{Aff}(\mathbb{R})$ -Lie system. By applying Theorem 3.3, we see that the Lie symmetries of (5.32) of the form

$$Y = f_0(t) \frac{\partial}{\partial t} + f_1(t)X_1 + f_2(t)X_2$$

are determined by the first-order system (5.28) with $a(t) = 1$ and $b(t) = -a_2(t)$, i.e.,

$$\begin{cases} \frac{df_0}{dt} = b_0(t), \\ \frac{df_1}{dt} = b_0(t) - a_2(t)f_1 - f_2, \\ \frac{df_2}{dt} = -f_0 \frac{da_2}{dt} - b_0(t)a_2(t), \end{cases} \tag{5.33}$$

which is associated to the t -dependent vector field

$$\Gamma_X^{BD} = b_0(t)Z_0 - \frac{da_2}{dt}W_2 + b_0(t)Z_1 - a_2(t)b_0(t)Z_2 + Y_1 - a_2(t)Y_2, \tag{5.34}$$

where the vector fields $Z_0, Z_1, Z_2, Y_1, Y_2, W_2$ are those detailed in (5.3) and have the commutation relations (5.29). Hence, these vector fields span a Lie algebra (5.30). Therefore, we can obtain the Lie symmetries for this system of the form (5.27) by substituting $a(t) = 1$ and $b(t) = -a_2(t)$ into (5.31).

5.4. A Painlevé–Ince equation

Let us study the Lie symmetries of the Painlevé–Ince equation [1,24,26]

$$\frac{d^2x}{dt^2} = -3x \frac{dx}{dt} - x^3. \tag{5.35}$$

This differential equation has been widely studied in many works (see [11] and references therein) due to its interesting properties and applications. From the point of view of the theory of Lie systems, it is a HODE Lie system, i.e., by adding the new variable $v \equiv dx/dt$, we obtain a system

$$\begin{cases} \frac{dx}{dt} = v, \\ \frac{dv}{dt} = -3xv - x^3, \end{cases} \tag{5.36}$$

which can be viewed as an $\mathfrak{sl}(3, \mathbb{R})$ -Lie system with a Vessiot–Gulberg Lie algebra V spanned by the vector fields (see [11])

$$\begin{aligned} X_1 &= v \frac{\partial}{\partial x} - (3xv + x^3) \frac{\partial}{\partial v}, & X_2 &= \frac{\partial}{\partial v}, \\ X_3 &= -\frac{\partial}{\partial x} + 3x \frac{\partial}{\partial v}, & X_4 &= x \frac{\partial}{\partial x} - 2x^2 \frac{\partial}{\partial v}, \\ X_5 &= (v + 2x^2) \frac{\partial}{\partial x} - x(v + 3x^2) \frac{\partial}{\partial v}, & X_6 &= 2x(v + x^2) \frac{\partial}{\partial x} + 2(v^2 - x^4) \frac{\partial}{\partial v}, \\ X_7 &= \frac{\partial}{\partial x} - x \frac{\partial}{\partial v}, & X_8 &= 2x \frac{\partial}{\partial x} + 4v \frac{\partial}{\partial v}. \end{aligned} \tag{5.37}$$

As a consequence, we can apply our theory to obtain Lie symmetries of this HODE Lie system. Moreover, as (5.36) determines the integral curves of X_1 , we can search for a vector field within V commuting with X_1 . Since $[X_1, X_6] = 0$ (see [11]), we obtain that X_6 is a Lie symmetry for (5.36) and a non-local Lie symmetry for (5.35).

6. A class of Lie symmetries for PDE Lie systems

Let us generalize in this section some of the previous results to the PDE Lie systems. We hereafter denote $t = (t_1, \dots, t_s) \in \mathbb{R}^s$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Theorem 6.1. *Let V be a finite-dimensional real Lie algebra of vector fields on \mathbb{R}^n with a basis X_1, \dots, X_r . Given a PDE Lie system*

$$\frac{\partial x_i}{\partial t_l} = \sum_{\alpha=1}^r b_{\alpha l}(t) X_{\alpha}(x), \quad l = 1, \dots, s, \quad i = 1, \dots, n, \tag{6.1}$$

then

$$Y(t, x) = \sum_{\beta=1}^r f_{\beta}(t) X_{\beta}(x) \tag{6.2}$$

is a Lie symmetry of (6.1) if and only if $[\tilde{X}_l^i, Y] = 0$ for every l , where \tilde{X}_l^i is the l -autonomization given by (2.7) of the t -dependent vector field

$$X_l(t, x) \equiv \sum_{\alpha=1}^r b_{\alpha l}(t) X_{\alpha}(x), \quad l = 1, \dots, s.$$

Proof. The coordinate systems $\{t_1, \dots, t_s\}$ on \mathbb{R}^s and $\{x_1, \dots, x_n\}$ on \mathbb{R}^n induce a coordinate system on the fiber bundle $J^1\pi \simeq \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^{ns}$ with respect to the projection $\pi : (t_l, x_j) \in \mathbb{R}^s \times \mathbb{R}^n \mapsto (t_l) \in \mathbb{R}^s$ of the form $t_l, x_j, x_{j,l}$, with $1 \leq j \leq n$ and $1 \leq l \leq s$. A vector field $Y = \sum_{k=1}^n \eta_k(t, x) \partial/\partial x_k$, $t \in \mathbb{R}^s$ and $x \in \mathbb{R}^n$, is a Lie symmetry of (6.1) if and only if

$$\widehat{Y} F_l^i = 0, \quad i = 1, \dots, n, \quad l = 1, \dots, s$$

on the submanifold $\mathcal{S} = \cap_{i=1}^n \cap_{l=1}^s (F_l^i)^{-1}(0)$, with $\widehat{Y} : J^1\pi \rightarrow T(J^1\pi)$ being the prolongation of Y , namely

$$\widehat{Y} = \sum_{k=1}^n \left[\eta_k \frac{\partial}{\partial x_k} + \sum_{q=1}^s \left(\frac{\partial \eta_k}{\partial t_q} + \sum_{j=1}^n \frac{\partial \eta_k}{\partial x_j} x_{j,q} \right) \frac{\partial}{\partial x_{k,q}} \right],$$

and

$$F_l^i = x_{i,l} - \sum_{\alpha=1}^r b_{\alpha l}(t) X_{\alpha}^i(x), \quad i = 1, \dots, n, \quad l = 1, \dots, s,$$

$$X_\alpha = \sum_{i=1}^n X_\alpha^i(x) \frac{\partial}{\partial x_i}, \quad \alpha = 1, \dots, r.$$

By assumption, $\eta_k(t, x) = \sum_{\beta=1}^r f_\beta(t) X_\beta^k(x)$ for $k = 1, \dots, n$. So,

$$\hat{Y} = \sum_{k=1}^n \sum_{\beta=1}^r \left[f_\beta(t) X_\beta^k \frac{\partial}{\partial x_k} + \sum_{q=1}^s \left(\frac{\partial f_\beta(t)}{\partial t_q} X_\beta^k + \sum_{j=1}^n f_\beta(t) \frac{\partial X_\beta^k}{\partial x_j} x_{j,q} \right) \frac{\partial}{\partial x_{k,q}} \right].$$

Substituting this in $\hat{Y}F_l^i = 0$, we obtain

$$\begin{aligned} & \sum_{k=1}^n \sum_{\beta=1}^r \left[f_\beta(t) X_\beta^k \left(- \sum_{\alpha=1}^r b_{\alpha l}(t) \frac{\partial X_\alpha^i}{\partial x_k} \right) + \sum_{q=1}^s \left(\frac{\partial f_\beta(t)}{\partial t_q} X_\beta^k + \sum_{j=1}^n f_\beta(t) \frac{\partial X_\beta^k}{\partial x_j} x_{j,q} \right) \delta_q^l \delta_k^i \right] \\ & = \sum_{\beta=1}^r \left[- \sum_{k=1}^n \sum_{\alpha=1}^r f_\beta(t) b_{\alpha l}(t) X_\beta^k \frac{\partial X_\alpha^i}{\partial x_k} + \frac{\partial f_\beta(t)}{\partial t_l} X_\beta^i + \sum_{j=1}^n f_\beta(t) \frac{\partial X_\beta^i}{\partial x_j} x_{j,l} \right] = 0. \end{aligned}$$

Restricting the above expression to the submanifold $\mathcal{S} = \cap_{i=1}^n \cap_{l=1}^s (F_l^i)^{-1}(0)$ and renaming indexes appropriately, we obtain

$$\sum_{\beta=1}^r \left[\frac{\partial f_\beta(t)}{\partial t_l} X_\beta^i + \sum_{k=1}^n \sum_{\alpha=1}^r \left(f_\beta(t) \frac{\partial X_\beta^i}{\partial x_k} b_{\alpha l}(t) X_\alpha^k - f_\beta(t) b_{\alpha l}(t) X_\beta^k \frac{\partial X_\alpha^i}{\partial x_k} \right) \right] = 0.$$

Hence,

$$\sum_{\beta=1}^r \left[\frac{\partial f_\beta(t)}{\partial t_l} X_\beta^i + \sum_{\gamma=1}^r \sum_{k=1}^n f_\beta(t) b_{\gamma l}(t) \left(\frac{\partial X_\beta^i}{\partial x_k} X_\gamma^k - X_\beta^k \frac{\partial X_\gamma^i}{\partial x_k} \right) \right] = 0,$$

whose right-hand side becomes, for each fixed l , the coefficients in the basis $\partial/\partial x^i$, with $i = 1, \dots, n$, of $[\bar{X}_l^i, Y]$. So, the above amounts to

$$\begin{aligned} [\bar{X}_l^i, Y] &= \left[\frac{\partial}{\partial t_l} + \sum_{k=1}^n \sum_{\gamma=1}^r b_{\gamma l}(t) X_\gamma^k \frac{\partial}{\partial x_k}, \sum_{i=1}^n \sum_{\beta=1}^r f_\beta(t) X_\beta^i \frac{\partial}{\partial x_i} \right] \\ &= \left[\frac{\partial}{\partial t_l} + \sum_{\gamma=1}^r b_{\gamma l}(t) X_\gamma, \sum_{\beta=1}^r f_\beta(t) X_\beta \right] = 0. \end{aligned}$$

Then, Y is a Lie symmetry of (6.1) if and only if the condition

$$[\bar{X}_l^i, Y] = 0 \tag{6.3}$$

is satisfied for $l = 1, \dots, s$. \square

Theorem 6.2. Given a Lie symmetry of the form (6.2) for the system (6.1), the coefficients $f_1(t), \dots, f_r(t)$ satisfy a PDE Lie system admitting a Vessiot–Guldberg Lie algebra $V^s \simeq V/Z(V)$, where we recall that V is a Vessiot–Guldberg Lie algebra for (6.1).

Proof. Let X_1, \dots, X_r be a basis for V with structure constants $c_{\alpha\beta\gamma}$. From Theorem 6.1, we have

$$\begin{aligned} [\bar{X}_l^i, Y] &= \left[\frac{\partial}{\partial t_l} + \sum_{\alpha=1}^r b_{\alpha l}(t) X_\alpha, \sum_{\delta=1}^r f_\delta(t) X_\delta \right] = \sum_{\delta=1}^r \left(\frac{\partial f_\delta}{\partial t_l} X_\delta + \sum_{\alpha=1}^r b_{\alpha l}(t) f_\delta [X_\alpha, X_\delta] \right) \\ &= \sum_{\pi=1}^r \left(\frac{\partial f_\pi}{\partial t_l} + \sum_{\alpha=1}^r \sum_{\delta=1}^r b_{\alpha l}(t) f_\delta c_{\alpha\delta\pi} \right) X_\pi = 0. \end{aligned}$$

Since X_1, \dots, X_r are linearly independent over \mathbb{R} and the coefficients of the above expression are only t -dependent, we get that the above amounts to

$$\frac{\partial f_\pi}{\partial t_l} = \sum_{\alpha,\delta=1}^r b_{\alpha l}(t) f_\delta c_{\delta\alpha\pi}, \quad \pi = 1, \dots, r, \quad l = 1, \dots, s. \tag{6.4}$$

To prove that this is a PDE Lie system, we define the vector fields

$$Y_\alpha = \sum_{\delta,\pi=1}^r c_{\delta\alpha\pi} f_\delta \frac{\partial}{\partial f_\pi}, \quad \alpha = 1, \dots, r.$$

We have already proved that $[Y_\alpha, Y_\beta] = \sum_{\delta=1}^r c_{\alpha\beta\delta} Y_\delta$ in [Theorem 3.3](#). So, these vector fields span a Lie algebra isomorphic to $V/Z(V)$ (for a proof of this fact follow the same line of reasoning as in [Theorem 3.3](#)). In terms of these vector fields, we see that [\(6.4\)](#) is related to the t -dependent vector fields $X_l(t, x) = \sum_{\alpha=1}^r b_{\alpha l}(t) X_\alpha(x)$, with $l = 1, \dots, s$. Additionally, to be a PDE Lie system, the above system [\(6.4\)](#) must satisfy the condition

$$\sum_{\alpha, \mu=1}^r \left(\frac{\partial b_{\alpha\pi}}{\partial \kappa} - \frac{\partial b_{\alpha\kappa}}{\partial \pi} + \sum_{\delta, \epsilon=1}^r b_{\delta\kappa} b_{\epsilon\pi} c_{\delta\epsilon\alpha} \right) f_\mu c_{\mu\alpha\sigma} = 0, \quad \kappa \neq \pi = 1, \dots, r, \quad \sigma = 1, \dots, r. \tag{6.5}$$

The expression in brackets vanishes due to the integrability condition for system [\(6.1\)](#). Hence, [\(6.4\)](#) is a PDE Lie system. We call it the *symmetry system* for [\(6.1\)](#) relative to V . \square

Definition 6.3. Given a PDE Lie system X with a Vessiot–Guldberg Lie algebra V , we call S_X^V the space of Lie symmetries of X that are also t -dependent vector fields taking values in V .

We can straightforwardly prove that the space S_X^V is a Lie algebra.

7. Lie symmetries for $\mathfrak{sl}(2, \mathbb{R})$ -PDE Lie systems

An $\mathfrak{sl}(2, \mathbb{R})$ -PDE Lie system is a PDE Lie system admitting a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Let us obtain the elements of S_X^V for this case.

Let us choose a basis of vector fields $\{X_1, X_2, X_3\}$ for V satisfying the same commutation relations as in [\(5.1\)](#). Let us write a general PDE Lie system whose autonomization for a fixed value l is

$$\bar{X}_l^l = \frac{\partial}{\partial t_l} + b_{1l}(t)X_1 + b_{2l}(t)X_2 + b_{3l}(t)X_3, \quad t = (t_1, \dots, t_s) \in \mathbb{R}^s, \quad 1 \leq l \leq s$$

and a certain type of possible Lie symmetry $Y = f_1(t)X_1 + f_2(t)X_2 + f_3(t)X_3$, where $f_1(t), f_2(t), f_3(t)$ are t -dependent functions to be determined by the symmetry condition [\(6.3\)](#). This leads us to the system of s first-order PDEs

$$\begin{cases} \frac{\partial f_1}{\partial t_l} = b_{2l}(t)f_1 - b_{1l}(t)f_2, \\ \frac{\partial f_2}{\partial t_l} = 2(b_{3l}(t)f_1 - b_{1l}(t)f_3), \\ \frac{\partial f_3}{\partial t_l} = b_{3l}(t)f_2 - b_{2l}(t)f_3, \end{cases} \tag{7.1}$$

with $l = 1, \dots, s$. Expressed in terms of t -dependent vector fields, for a fixed value of l

$$\Gamma_l^{\mathfrak{sl}(2, \mathbb{R})} = b_{1l}(t)Y_1 + b_{2l}(t)Y_2 + b_{3l}(t)Y_3, \quad l = 1, \dots, s,$$

where

$$Y_1 = -f_2 \frac{\partial}{\partial f_1} - 2f_3 \frac{\partial}{\partial f_2}, \quad Y_2 = f_1 \frac{\partial}{\partial f_1} - f_3 \frac{\partial}{\partial f_3}, \quad Y_3 = 2f_1 \frac{\partial}{\partial f_2} + f_2 \frac{\partial}{\partial f_3} \tag{7.2}$$

close the commutation relations in [\(5.1\)](#). So, they span a Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

7.1. The partial Riccati equation

Let us consider the partial Riccati equation, i.e., the PDE system

$$\frac{\partial x}{\partial t_1} = b_{11}(t) + b_{21}(t)x + b_{31}(t)x^2, \quad \frac{\partial x}{\partial t_2} = b_{12}(t) + b_{22}(t)x + b_{32}(t)x^2, \tag{7.3}$$

with the t -dependent coefficients satisfying the appropriate integrability condition [\(6.5\)](#). Such systems appear in the study of WZNW equations and multidimensional Toda systems [\[20\]](#). Observe that the partials $\partial x / \partial t_1$ and $\partial x / \partial t_2$ are related to the t -dependent vector fields $X_{t_1}^{\text{pRic}} = b_{11}(t)X_1^{\text{pRic}} + b_{21}(t)X_2^{\text{pRic}} + b_{31}(t)X_3^{\text{pRic}}$ and $X_{t_2}^{\text{pRic}} = b_{12}(t)X_1^{\text{pRic}} + b_{22}(t)X_2^{\text{pRic}} + b_{32}(t)X_3^{\text{pRic}}$, with

$$X_1^{\text{pRic}} = \frac{\partial}{\partial x}, \quad X_2^{\text{pRic}} = x \frac{\partial}{\partial x}, \quad X_3^{\text{pRic}} = x^2 \frac{\partial}{\partial x}$$

satisfying the commutation relations [\(5.1\)](#). That is, the vector fields $\langle X_1^{\text{pRic}}, X_2^{\text{pRic}}, X_3^{\text{pRic}} \rangle$ span a Vessiot–Guldberg Lie algebra $V^{\text{pRic}} \simeq \mathfrak{sl}(2, \mathbb{R})$. Since we assume that the functions $b_{ij}(t)$ with $i = 1, 2, 3$ and $j = 1, 2$ satisfy [\(6.5\)](#), we get that [\(7.3\)](#) is a PDE Lie system.

Let us look for Lie symmetries of the form $Y = f_1(t)X_1^{\text{pRic}} + f_2(t)X_2^{\text{pRic}} + f_3(t)X_3^{\text{pRic}}$ for (7.3). In view of Theorem 6.2, such Lie symmetries are solutions of the system of PDEs

$$\begin{cases} \frac{\partial f_1}{\partial t_j} = b_{2j}(t)f_1 - b_{1j}(t)f_2, \\ \frac{\partial f_2}{\partial t_j} = 2(b_{3j}(t)f_1 - b_{1j}(t)f_3), \\ \frac{\partial f_3}{\partial t_j} = b_{3j}(t)f_2 - b_{2j}(t)f_3, \end{cases} \tag{7.4}$$

with $j = 1, 2$. This resulting system can be interpreted in terms of the t -dependent vector fields $\Gamma_j^{\text{pRic}} = b_{1j}(t)Y_1 + b_{2j}(t)Y_2 + b_{3j}(t)Y_3$, with $j = 1, 2$ and (7.2). These vector fields have the same structure constants as the $X_1^{\text{pRic}}, X_2^{\text{pRic}}, X_3^{\text{pRic}}$. Therefore, (7.3) is a PDE Lie system with a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

7.2. Flat \mathfrak{g} -valued forms and their generators

Let G be a Lie group with Lie algebra \mathfrak{g} and let $L_g: G \rightarrow G$ be the left-action of G on G . We write ω for a one-form on an n -dimensional manifold N taking values in \mathfrak{g} and $\varphi: N \rightarrow G$ for a function on N taking values in G . Consider the system of PDEs

$$\omega = L_{\varphi^{-1}*}d\varphi$$

determining the hereafter called generator φ of ω , where

$$d\omega + \omega \wedge \omega = 0, \tag{7.5}$$

and $\omega \wedge \omega$ stands for the wedge product of the Lie algebra-valued ω with itself. Equations of this type appear for instance in the study multidimensional Riccati type equations and Lie algebra valued flat connection forms.

Taking a local basis of coordinates $\{x_1, \dots, x_n\}$ on N and writing $\omega = \sum_{\mu=1}^n \omega_\mu(x)dx_\mu$, where $\omega_\mu : N \rightarrow \mathfrak{g}$, we get

$$\frac{\partial \varphi}{\partial x_\mu} = L_{\varphi*}\omega_\mu, \quad \mu = 1, \dots, n.$$

Choosing a basis v_1, \dots, v_r for \mathfrak{g} , we can write $\omega_\mu = \sum_{\alpha=1}^r f_{\alpha\mu}(x)v_\alpha$ for $\mu = 1, \dots, n$ and certain functions $f_{\alpha\mu}(x)$ with $\alpha = 1, \dots, r$ on N . Hence,

$$\frac{\partial \varphi}{\partial x_\mu} = L_{\varphi*} \left(\sum_{\alpha=1}^r f_{\alpha\mu}(x)v_\alpha \right) = \sum_{\alpha=1}^r f_{\alpha\mu}(x)L_{\varphi*}v_\alpha, \quad \mu = 1, \dots, n.$$

Note that $L_{\varphi*}v_\alpha$ can be considered as the value at $\varphi \in G$ of the left-invariant vector field on G given by $X_\alpha^L(\varphi) = L_{\varphi*}v_\alpha$. In consequence, we have

$$\frac{\partial \varphi}{\partial x_\mu} = \sum_{\alpha=1}^r f_{\alpha\mu}(x)X_\alpha^L(\varphi), \quad \mu = 1, \dots, n. \tag{7.6}$$

Additionally, in view of (7.5), we have that

$$\frac{\partial f_{\alpha\mu}}{\partial x_\beta} - \frac{\partial f_{\alpha\beta}}{\partial x_\mu} + \sum_{\pi, \delta=1}^r f_{\pi\beta}f_{\delta\mu}c_{\pi\delta\alpha} = 0, \quad \mu \neq \beta = 1, \dots, n, \quad \alpha = 1, \dots, n,$$

and hence, (7.6) becomes a PDE Lie system.

In terms of the basis X_1^L, \dots, X_r^L for \mathfrak{g} , which have structure constants $c_{\alpha\beta\gamma}$, we see that the symmetry system for (7.6) reads as (6.4).

8. Conclusions and Outlook

We have constructed a natural family of Lie symmetries for Lie systems that can be described, studied and derived through the theory of Lie systems. In particular, we have shown that these Lie symmetries can be obtained by solving another Lie system possessing a Vessiot–Guldberg Lie algebra related to the Vessiot–Guldberg Lie algebra of the Lie system under study. Our results have been generalized to the realm of HODEs and PDE Lie systems. Our procedures enable us to find Lie symmetries of isomorphic (PDE) Lie systems simultaneously.

To illustrate our theory, we have studied Lie symmetries for some systems of differential equations of physical and mathematical relevancy. As a result, we have retrieved and generalized several results concerning Lie symmetries of several types of known and new Riccati equations from a more geometrical point of view. Moreover, we have shown that autonomous systems can also be studied through Lie systems.

In the future, we aim to introduce a more general perspective on the quaternionic Riccati equation by imposing that the coefficients take values in the quaternionic field. Additionally, we aim to study the generalization of the quaternionic Riccati equation to other types of algebras and to study their geometric properties. Moreover, we also plan to perform a deeper study on PDE Lie systems. All these things shall be included in forthcoming papers.

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