On Lie systems and Kummer–Schwarz equations

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Abstract

A *Lie system* is a system of first-order differential equations admitting a *superposition rule*, i.e., a map that expresses its general solution in terms of a generic family of particular solutions and certain constants. In this work, we use the geometric theory of Lie systems to prove that the explicit integration of second- and third-order Kummer–Schwarz equations is equivalent to obtaining a particular solution of a Lie system on $SL(2,\mathbb{R})$. This same result can be extended to Riccati, Milne–Pinney and other related equations. We demonstrate that all the above-mentioned equations associated with exactly the same Lie system on $SL(2,\mathbb{R})$ can be integrated simultaneously. This retrieves and generalizes in a unified and simpler manner previous results appearing in the literature. As a byproduct, we recover various properties of the *Schwarzian derivative*.

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1 Introduction

Geometric techniques have been proved to be a very successful approach to solve differential equations, leading to methods of key importance in physics and mathematics, such as: Lie symmetries, the Painlevé method, and Lax pairs [1, 2, 3, 4].

In this work, we focus on the geometric theory of Lie systems [5, 6, 7, 8, 9, 10, 11, 12]. Lie systems have lately attracted some attention owing to their numerous applications and properties [13]. For instance, they have been employed to study the integrability of Riccati and matrix Riccati equations [14, 15, 16], control and Floquet theory [17, 18, 19, 20], and other equations appearing in classical and quantum mechanics [13]. Furthermore, their generalizations have led to the geometric investigation of stochastic equations [21], superequations [22], and other topics [23, 24].

More specifically, we analyze second- and third-order Kummer–Schwarz equations [25, 26, 27, 28] – henceforth KS-2 and KS-3 equations – with the aid of the theory of Lie systems. The mathematical relevance of these equations resides in their close connection with the Kummer's problem [28, 29, 30], the study of homogeneous systems of second-order differential equations [31], the Schwarzian derivative [32], and other related themes [27, 28, 29, 33, 34, 35, 36]. Moreover, some interest has been focused on the study of solutions of KS-2 and KS-3 equations, which have been analyzed in several manners in the literature: e.g., through non-local transformations or in terms of solutions to other differential equations [25, 26, 27, 29, 30]. From a physical viewpoint, KS-2 and KS-3 equations occur in the study of Milne–Pinney equations, Riccati equations, and time-dependent frequency harmonic oscillators [24, 25, 26, 37], which are of certain relevancy in two-body problems [38, 39], quantum mechanics [40, 41], classical mechanics [42], etcetera [43].

First, we show that KS-2 and KS-3 equations can be studied through two $\mathfrak{sl}(2,\mathbb{R})$–Lie systems [44], i.e., Lie systems that describe the integral curves of a t-dependent vector field taking values in a Lie algebra of vector fields -a so-called *Vessiot–Guldberg Lie algebra*- isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. This new result slightly generalizes previous findings about these equations [24].

Afterwards, we obtain two Lie group actions whose fundamental vector fields correspond with those of the above-mentioned Vessiot–Guldberg Lie algebras. These actions allow us to prove that the explicit integration of KS-2 and KS-3 equations is equivalent to working out a particular solution of a Lie system on $SL(2,\mathbb{R})$. Further, we will see that Riccati and Milne–Pinney equations exhibit similar features.

We show that the knowledge of the general solution of any of the abovementioned equations allows us to solve simultaneously any other related to the same equation on $SL(2,\mathbb{R})$. This fact provides a new powerful and general way of linking solutions of these equations, which were previously known to be related through *ad hoc* expressions in certain cases [26, 37]. Additionally, our approach can potentially be extended to other $\mathfrak{sl}(2,\mathbb{R})$-Lie systems [13].

Subsequently, we derive a superposition rule for certain Lie systems associated with a relevant family of KS-3 equations of the form $\{x,t\} = 2b_1(t)$, with $b_1(t)$ being an arbitrary function of t and $\{x,t\}$ the Schwarzian derivative of a function $x(t)$ with respect to t [27, 32]. This permits us to recover some features of these remarkable equations [32] and the Schwarzian derivatives. In particular, we prove that the general solution of $\{x,t\} = 2b_1(t)$ can be described by means of an expression that depends on a particular solution and several constants related to the initial conditions, i.e., a *basic superposition rule for higher-order differential equations* [24]. This enables us to solve some relevant cases of these equations. Additionally, our basic superposition rule allows us to retrieve some known symmetries of Schwarzian derivatives [32].

The direct Lie–Scheffers theorem [45] states that every Lie system possesses a *mixed superposition rule*, i.e., a type of functions that allows us to express its general solution in terms of particular solutions of (possibly different) systems of first-order differential equations and some constants [45]. We here derive a mixed superposition rule to investigate general solutions of KS-2 equations in terms of t-dependent frequency harmonic oscillators. This mixed superposition rule generalizes a previous result described by Berkovich [30]. It is also remarkable that mixed superposition rules can be applied to describe general solutions of second- and third-order Kummer–Schwarz equations in terms of other $\mathfrak{sl}(2,\mathbb{R})$-Lie systems. An example is to reproduce expressions describing the general solutions of $\{x,t\} = 2b_1(t)$ in terms of particular solutions of Riccati equations or certain time-dependent frequency harmonic oscillators

[46].

The structure of the paper goes as follows. Section II is devoted to surveying several concepts used throughout the paper. In Section III and IV, we address the analysis of KS-2 and KS-3 equations and show how their integration can be reduced to solving certain Lie systems on $SL(2,\mathbb{R})$. Section V is dedicated to the analysis of connections of second- and third-order Kummer–Schwarz equations with other equations. In Section VI we accomplish a new approach to the Schwarzian derivative based upon our results. Subsequently, we provide a new superposition rule for KS-3 equations in Section VII. To conclude, we resume our results and describe the future work to do in Section VIII.

2 Fundamentals

For simplicity, we hereafter assume all mathematical objects, e.g., vector fields and superposition rules, to be real, smooth, and globally defined on vector spaces. In this manner, we highlight the key points of our presentation by omitting the analysis of minor technical problems (see [11, 12, 47] for details).

2.1 On t-dependent vector fields

The geometrical study of Lie systems is based upon the notion of *t-dependent vector fields* [48]. Given the tangent bundle projection $\tau : \mathrm{T}\mathbb{R}^n \to \mathbb{R}^n$ and the projection $\pi_2 : (t,x) \in \mathbb{R}\times\mathbb{R}^n \mapsto x \in \mathbb{R}^n$, a *$t$-dependent vector field* X on $\mathbb{R}^n$ is a map $X : (t,x) \in \mathbb{R}\times\mathbb{R}^n \mapsto X(t,x) \in \mathrm{T}\mathbb{R}^n$ satisfying that $\tau \circ X = \pi_2$. This condition implies that X can be considered as a family $\{X_t\}_{t\in\mathbb{R}}$ of vector fields $X_t : x \in \mathbb{R}^n \mapsto X_t(x) = X(t,x) \in \mathrm{T}_x\mathbb{R}^n$ and vice versa [13].

As standard vector fields, t-dependent vector fields also admit integral curves. We call *integral curve* of X an integral curve $\gamma : \mathbb{R} \to \mathbb{R}\times\mathbb{R}^n$ of the *suspension* of X, i.e., the vector field on $\mathbb{R}\times N$ given by $\partial/\partial t + X(t,x)$ [48].

From a modern geometric viewpoint, every system of first-order differential equations

$$\frac{dx^i}{dt} = X^i(t,x), \qquad i = 1,\ldots,n, \tag{1}$$

can be associated with the unique t-dependent vector field on $\mathbb{R}^n$, namely,

$$X(t,x) = \sum_{i=1}^{n} X^i(t,x)\frac{\partial}{\partial x^i}, \tag{2}$$

whose integral curves are (up to an appropriate reparametrization) of the form $(t,x(t))$, with $x(t)$ being a solution of system (1). Conversely, every t-dependent vector field (2) determines a unique system of first-order differential equations, the so-called *associated system*, determining its integral curves of the form $(t,x(t))$. This justifies to denote by X both a t-dependent vector field and its associated system.

2.2 Lie systems and superposition rules

A *superposition rule* for a system X on $\mathbb{R}^n$ is a map $\Phi : (\mathbb{R}^n)^m \times \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$u = \Phi(u_{(1)}, \ldots, u_{(m)}; k_1, \ldots, k_n),$$

such that the general solution $x(t)$ of X can be written as

$$x(t) = \Phi\left(x_{(1)}(t), \ldots, x_{(m)}(t); k_1, \ldots, k_n\right), \tag{3}$$

with $x_{(1)}(t), \ldots, x_{(m)}(t)$ being a generic family of particular solutions and $k_1, \ldots, k_n$ being a set of constants related to the initial conditions of the system.

The characterization of systems of first-order differential equations admitting a superposition rule was obtained by Lie [5]. Its result, the nowadays called *Lie–Scheffers Theorem*, is the cornerstone of the theory of Lie systems [12, 49] and related theories [13, 45].

The Lie–Scheffers theorem states that a system X possesses a superposition rule if and only if

$$X_t = \sum_{\alpha=1}^{r} b_\alpha(t) Y_\alpha, \tag{4}$$

for a certain family $Y_1, \ldots, Y_r$ of vector fields spanning an r-dimensional real Lie algebra, the so-called associated *Vessiot–Guldberg Lie algebra*, and t-dependent functions $b_1(t), \ldots, b_r(t)$.

Every Lie system X associated with a Vessiot–Guldberg Lie algebra V gives rise to a (generally local) Lie group action $\varphi : G \times \mathbb{R}^n \to \mathbb{R}^n$ whose fundamental vector fields are the elements of V and such that $T_eG \simeq V$ with e being the neutral element of G [50]. This action allows us to write the general solution of X, which can be assumed to be of the form (4), as

$$x(t) = \varphi(g_1(t), x_0), \qquad x_0 \in \mathbb{R}^n, \tag{5}$$

with $g_1(t)$ being a particular solution of

$$\frac{dg}{dt} = -\sum_{\alpha=1}^{r} b_\alpha(t) Y^R_\alpha(g), \tag{6}$$

where $Y^R_1, \ldots, Y^R_r$ is a certain basis of right-invariant vector fields on G such that $Y^R_\alpha(e) = \mathrm{a}_\alpha \in T_eG$, with $\alpha = 1, \ldots, r$, and each a_α is the element of T_eG associated to the fundamental vector field Y_α (see [11] for details).

Since $Y^R_1, \ldots, Y^R_r$ span a finite-dimensional real Lie algebra, the Lie–Scheffers Theorem guarantees that (6) admits a superposition rule and becomes a Lie system. Indeed, as the right-hand side of (6) is invariant under right-translations, its general solution can be brought into the form

$$g(t) = R_{g_0} g_1(t), \qquad g_0 \in G, \tag{7}$$

where $g_1(t)$ is a particular solution of (6) and R_{g_0}, with $g_0 \in G$, is the map $R_{g_0} : g' \in G \mapsto g' \cdot g_0 \in G$ [11]. In other words, (6) admits a superposition rule.

Finally, given the general solution of X, the solution $g_1(t)$ of the associated (6) with $g_1(0) = e$ can be characterized as the unique solution to the algebraic system $x_p(t) = \varphi(g_1(t), x_p(0))$, where $x_p(t)$ ranges over a "sufficient large set" of particular solutions of X [10].

2.3 Mixed superposition rules

Despite its theoretical relevance, expression (5) only becomes useful to obtain explicitly the general solution of X in terms of a particular solution of (6) or vice versa, provided the explicit form of φ is known. Unfortunately, this is usually a complicated task as, for instance, in the case of most autonomous Lie systems [47]. Nevertheless, expression (5) is also interesting for constituting a particular example of a *mixed superposition rule*, which generalizes the notion of superposition rules [45].

A *mixed superposition rule* for a system X is a $(m+1)$-tuple $(\Phi, X_{(1)}, \ldots, X_{(m)})$ consisting of a function $\Phi : \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m} \times \mathbb{R}^n \to \mathbb{R}^n$ and a series of systems $X_{(a)}$ on $\mathbb{R}^{n_a}$, with $a = 1, \ldots, m$, such that the general solution $x(t)$ of X can be cast in the form

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t); k_1, \ldots, k_n), \tag{8}$$

where $x_{(1)}(t), \ldots, x_{(m)}(t)$ is a generic family of particular solutions of $X_{(1)}, \ldots, X_{(m)}$, respectively, and $k_1, \ldots, k_n$ are real constants.

The *direct Lie–Scheffers Theorem* [45] states that a system X admits a mixed superposition rule if and only if X is a Lie system. Although this restricts the use of mixed superposition rules to Lie systems, the notion is still relevant, as mixed superposition rules are in general more versatile and easier to derive than standard ones [45].

Let us now describe a procedure to derive mixed superposition rules. This method is based on the so-called direct product of t-dependent vector fields and the geometrical characterization of mixed superposition rules as foliations (we refer to [13, 45, 51] for details and examples).

Given a certain family $X_{(0)}, \ldots, X_{(m)}$ of t-dependent vector fields defined, respectively, on $\mathbb{R}^{n_0}, \ldots, \mathbb{R}^{n_m}$, their *direct product* (or *direct prolongation*) $Z \equiv X_{(0)} \times \cdots \times X_{(m)}$ is the unique t-dependent vector field Z on $\mathbb{R}^{n_0} \times \ldots \times \mathbb{R}^{n_m}$ such that $\mathrm{pr}_{a*} Z_t = (X_{(a)})_t$, with $\mathrm{pr}_a : (x_{(0)}, \ldots, x_{(m)}) \in \mathbb{R}^{n_0} \times \ldots \times \mathbb{R}^{n_m} \mapsto x_{(a)} \in \mathbb{R}^{n_a}$ for $a = 0, \ldots, m$ and all $t \in \mathbb{R}$.

It can be proved that every mixed superposition rule $(\Phi, X_{(1)}, \ldots, X_{(m)})$ for a system X on $\mathbb{R}^n$ gives rise to an n-codimensional foliation $\mathcal{F}$ on $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m} \times \mathbb{R}^n$ whose leaves $\mathcal{F}_k$, with $k = (k_1, \ldots, k_n) \in \mathbb{R}^n$, are of the form

$$\mathcal{F}_k = \{(x_{(1)}, \ldots, x_{(m)}, x) | x = \Phi(x_{(1)}, \ldots, x_{(m)}; k)\}.$$

Such leaves project diffeomorphically onto $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m}$ via

$$\begin{array}{rccc} \mathrm{pr}: & \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m} \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_m} \\ & (x_{(1)}, \ldots, x_{(m)}, x) & \mapsto & (x_{(1)}, \ldots, x_{(m)}) \end{array} \tag{9}$$

and the vector fields $\{(X_{(1)} \times \ldots \times X_{(m)} \times X)_t\}_{t \in \mathbb{R}}$ are tangent to them. Conversely, it is known that a foliation of the above type also gives rise to a mixed superposition rule $(\Phi, X_{(1)}, \ldots, X_{(m)})$ for X [45].

We turn to describing a procedure to construct a foliation of the above type and, from it, a mixed superposition rule for a system X. In view of the direct Lie–Scheffers theorem, this is only possible if X is a Lie system. If so, assume V to be an associated Vessiot–Guldberg Lie algebra. Take a basis $Y_1, \dots, Y_r$ of V. Determine a family $V_{(1)}, \dots, V_{(m)}$ of Lie algebras of vector fields on $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$, respectively, isomorphic to V and admitting a series of bases

$$Y_1^{(a)}, \dots, Y_r^{(a)} \in V^{(a)}, \qquad a = 1, \dots, m,$$

that satisfy the same commutating relations as $Y_1, \dots, Y_r$ and such that the vector fields $\widehat{Y}_\alpha \equiv Y_\alpha^{(1)} \times \dots \times Y_\alpha^{(m)}$, with $\alpha = 1, \dots, r$, are linearly independent at a generic point (note that we have $r \le \sum_{a=1}^m n_a$). It is important to remark that these bases can easily be found in the literature of Lie systems [13], which facilitates the derivation of mixed superposition rules. Indeed, it will be posteriorly shown that bases of this type appear naturally throughout this work.

From the above bases, construct a set of vector fields $\widetilde{Y}_\alpha = \widehat{Y}_\alpha \times Y_\alpha$ on $\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m} \times \mathbb{R}^n$, with $\alpha = 1, \dots, r$. These vector fields span a real Lie algebra $\widetilde{V}$ of dimension $r \le \sum_{a=1}^m n_a$. Furthermore, their elements span an integrable distribution $\mathcal{D}$ on $\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m} \times \mathbb{R}^n$ of rank r of the form

$$\mathcal{D}_\xi = \{Z(\xi) \mid Z \in \widetilde{V}\} \subset \mathrm{T}_\xi(\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m} \times \mathbb{R}^n),$$

where $Z(\xi)$ is the value of the vector field Z at ξ. Besides, $\mathcal{D}$ projects under (9) onto an integrable distribution on $\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m}$ spanned by the vector fields $\widehat{Y}_1, \dots, \widehat{Y}_r$ and $\mathrm{pr}_{*\xi} : \mathcal{D}_\xi \subset \mathrm{T}_\xi(\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m} \times \mathbb{R}^n) \to \mathrm{T}_{\mathrm{pr}(\xi)}(\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m})$ is an injective mapping for a generic $\xi \in \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m} \times \mathbb{R}^n$.

As $\mathcal{D}$ is an integrable distribution of rank $r \le \sum_{a=1}^m n_a$ defined on a manifold of dimension $\sum_{a=1}^m n_a + n$, it can be proved [13, 45] that there exist n functionally independent first-integrals $F_1, \dots, F_n$ for all vector fields of $\mathcal{D}$ satisfying that

$$\frac{\partial(F_1, \dots, F_n)}{\partial(x^1, \dots, x^n)} \neq 0, \qquad x \equiv (x^1, \dots, x^n) \in \mathbb{R}^n, \tag{10}$$

at a generic point of $\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m} \times \mathbb{R}^n$. Hence, given a point $(x_{(1)}, \dots, x_{(m)})$ and certain constants $(k_1, \dots, k_n)$, there exists a unique $x \in \mathbb{R}^n$ such that $F_i(x_{(1)}, \dots, x_{(m)}, x) = k_i$, for $i = 1, \dots, n$. For every $k = (k_1, \dots, k_n)$, the solutions of the equations $F_i = k_i$, with $i = 1, \dots, n$, define the points of a leaf $\mathcal{F}_k$, of an n-codimensional foliation $\mathcal{F}$ whose leaves project diffeomorphically onto $\mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m}$. Further, the vector fields $Z_t = \sum_{\alpha=1}^r b_\alpha(t) \widetilde{Y}_\alpha$, are tangent to the leaves of $\mathcal{F}$.

Note that $Z = X_{(1)} \times \dots X_{(m)} \times X$, with $X = \sum_{\alpha=1}^r b_\alpha(t) Y_\alpha$ and $X_{(a)} = \sum_{\alpha=1}^r b_\alpha(t) Y_\alpha^{(a)}$ for $a = 1, \dots, m$. In view of the characterization of mixed superposition rules as projectable foliations and using that the vector fields $\{Z_t\}_{t \in \mathbb{R}}$ are tangent to the leaves of $\mathcal{F}$, we obtain that $\mathcal{F}$ describes a mixed superposition rule for X depending on particular solutions of the systems $X_{(a)}$, with $a = 1, \dots, m$. In fact, by considering the equations $F_i = k_i$, with $i = 1, \dots, n$ and $k_1, \dots, k_n$ being certain real constants, we can describe the value of $x = (x^1, \dots, x^n)$ in terms of $k_1, \dots, k_n$ and $x_{(1)}, \dots, x_{(m)}$. It can be proved that the resulting expressions determine a map

$$\begin{array}{cccc} \Phi: & \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_m} \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & (x_{(1)}, \dots, x_{(m)}; k_1, \dots, k_n) & \mapsto & (x^1, \dots, x^n) \end{array} \tag{11}$$

that permits us to express the general solution $x(t)$ of X in the form (8) in terms of a generic family of particular solutions of the systems $X_{(1)}, \ldots, X_{(m)}$ and real constants $k_1, \ldots, k_n$, i.e., $(\Phi, X_{(1)}, \ldots, X_{(m)})$ is a mixed superposition rule for X.

Observe that if we additionally impose that $V_{(1)} = \ldots = V_{(m)} = V$ and choose the same basis $Y_1, \ldots, Y_r$ for all these Lie algebras, we recover the procedure for describing standard superposition rules detailed in [12]. Indeed, in this case, we obtain a family of vector fields $\widehat{Y}_\alpha = Y_\alpha \times \ldots \times Y_\alpha (m-\text{times})$, with $\alpha = 1, \ldots, r$, the so-called *diagonal prolongations* [12, 45] of Y_α to $\mathbb{R}^{mn}$. When m is such that the vector fields $\widehat{Y}_1, \ldots, \widehat{Y}_r$ are linearly independent (at a generic point), the determination of n common functionally independent first-integrals for $\widetilde{Y}_\alpha = \widehat{Y}_\alpha \times Y_\alpha$, with $\alpha = 1, \ldots, r$, satisfying (10) gives rise to a superposition rule for X depending on m particular solutions.

3 The second-order Kummer–Schwarz equations

Let us now turn to analyzing KS-2 equations. These equations take the form

$$\frac{d^2x}{dt^2} = \frac{3}{2x}\left(\frac{dx}{dt}\right)^2 - 2c_0x^3 + 2b_1(t)x, \tag{12}$$

with c_0 being a real constant and $b_1(t)$ a t-dependent function. KS-2 equations are a particular case of second-order Gambier equations [24, 25] and appear in the study of cosmological models [42]. In addition, their relations to other differential equations like Milne–Pinney equations [25], make them an alternative approach to the analysis of many physical problems [24, 26, 40].

Consider the first-order system

$$\begin{cases} \dfrac{dx}{dt} = v, \\ \dfrac{dv}{dt} = \dfrac{3}{2}\dfrac{v^2}{x} - 2c_0x^3 + 2b_1(t)x, \end{cases} \tag{13}$$

on $T\mathbb{R}_0$, with $\mathbb{R}_0 = \mathbb{R} - \{0\}$, obtained by adding the new variable $v \equiv dx/dt$ to the KS-2 equation (12). This system describes the integral curves of the t-dependent vector field

$$X_t = v\frac{\partial}{\partial x} + \left(\frac{3}{2}\frac{v^2}{x} - 2c_0x^3 + 2b_1(t)x\right)\frac{\partial}{\partial v} = M_3 + b_1(t)M_1, \tag{14}$$

where

$$M_1 = 2x\frac{\partial}{\partial v}, \qquad M_2 = x\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v}, \qquad M_3 = v\frac{\partial}{\partial x} + \left(\frac{3}{2}\frac{v^2}{x} - 2c_0x^3\right)\frac{\partial}{\partial v} \tag{15}$$

satisfy the commutation relations

$$[M_1, M_3] = 2M_2, \quad [M_1, M_2] = M_1, \quad [M_2, M_3] = M_3. \tag{16}$$

These vector fields span a three-dimensional real Lie algebra V of vector fields isomorphic to $\mathfrak{sl}(2,\mathbb{R})$[13, 24]. Hence, in view of (14) and the Lie–Scheffers Theorem, X admits a superposition rule and becomes a Lie system associated with a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, i.e., a $\mathfrak{sl}(2,\mathbb{R})$-Lie system.

As shown in Section 2, the knowledge of a Lie group action $\varphi_{2KS}: G\times \mathrm{T}\mathbb{R}_0 \to \mathrm{T}\mathbb{R}_0$ whose fundamental vector fields are V and $T_eG \simeq V$ allows us to express the general solution of X in the form (5), in terms of a particular solution of a Lie system (6) on G. Let us determine φ_{2KS} in such a way that our procedure can easily be extended to third-order Kummer–Schwarz equations.

Consider the basis of matrices of $\mathfrak{sl}(2,\mathbb{R})$

$$\mathrm{a}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{a}_2 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{a}_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{17}$$

satisfying the commutation relations

$$[\mathrm{a}_1,\mathrm{a}_3] = 2\mathrm{a}_2, \qquad [\mathrm{a}_1,\mathrm{a}_2] = \mathrm{a}_1, \qquad [\mathrm{a}_2,\mathrm{a}_3] = \mathrm{a}_3,$$

which match those satisfied by M_1, M_2 and M_3. So, the linear function $\rho : \mathfrak{sl}(2,\mathbb{R}) \to V$ mapping a_α into M_α, with $\alpha = 1,2,3$, is a Lie algebra isomorphism. If we consider it as an infinitesimal Lie group action, we can then ensure that there exists a local Lie group action $\varphi_{2KS}: SL(2,\mathbb{R})\times \mathrm{T}\mathbb{R}_0 \to \mathrm{T}\mathbb{R}_0$ obeying the required properties. In particular,

$$\frac{d}{ds}\varphi_{2KS}(\exp(-s\mathrm{a}_\alpha),\mathbf{t}_x) = M_\alpha(\varphi_{2KS}(\exp(-s\mathrm{a}_\alpha),\mathbf{t}_x)),$$

where $\mathbf{t}_x \equiv (x,v) \in \mathrm{T}_x\mathbb{R}_0 \subset \mathrm{T}\mathbb{R}_0$, $\alpha = 1,2,3$, and $s \in \mathbb{R}$. This condition determines the action on $\mathrm{T}\mathbb{R}_0$ of the elements of $SL(2,\mathbb{R})$ of the form $\exp(-s\mathrm{a}_\alpha)$, with $\alpha = 1,2,3$ and $s \in \mathbb{R}$. By integrating M_1 and M_2, we obtain

$$\begin{aligned} \varphi(\exp_{2KS}(-\lambda_1\mathrm{a}_1),\mathbf{t}_x) &= (x, v + 2x\lambda_1)\,, \\ \varphi(\exp_{2KS}(-\lambda_2\mathrm{a}_2),\mathbf{t}_x) &= \left(xe^{\lambda_2}, ve^{2\lambda_2}\right). \end{aligned} \tag{18}$$

Observe that M_3 is not defined on $\mathrm{T}_0\mathbb{R}$. So, its integral curves, let us say $(x(\lambda_3), v(\lambda_3))$, must be fully contained in either $\mathrm{T}\mathbb{R}^+$ or $\mathrm{T}\mathbb{R}^-$. These integral curves are determined by the system

$$\frac{dx}{d\lambda_3} = v, \qquad \frac{dv}{d\lambda_3} = \frac{3}{2}\frac{v^2}{x} - 2c_0x^3. \tag{19}$$

When $v \neq 0$, we obtain

$$\frac{dv^2}{dx} = \frac{3v^2}{x} - 4c_0x^3 \Longrightarrow v^2(\lambda_3) = x^3(\lambda_3)\Gamma - 4c_0x^4(\lambda_3),$$

for a real constant Γ. Hence, for each integral curve $(x(\lambda_3), v(\lambda_3))$, we have

$$\Gamma = \frac{v^2(\lambda_3) + 4c_0x^4(\lambda_3)}{x^3(\lambda_3)}.$$

Moreover, it easy to see that $d\Gamma/d\lambda_3 = 0$ not only for solutions of (19) with $v(\lambda_3) \neq 0$ for every λ_3, but for any solution of (19). Using the above results and (19), we see that

$$\frac{dx}{d\lambda_3} = \mathrm{sg}(v)\sqrt{\Gamma x^3 - 4c_0 x^4} \Rightarrow x(\lambda_3) = \frac{x(0)}{F_{\lambda_3}(x(0), v(0))}, \tag{20}$$

where sg is the well-known *sign function* and

$$F_{\lambda_3}(\mathbf{t}_x) = \left(1 - \frac{v\lambda_3}{2x}\right)^2 + c_0 x^2 \lambda_3^2.$$

Now, from (20) and taking into account the first equation within (19), it immediately follows that

$$\varphi_{2KS}(\exp(-\lambda_3 \mathrm{a}_3), \mathbf{t}_x) = \left(\frac{x}{F_{\lambda_3}(\mathbf{t}_x)}, \frac{v - \frac{v^2 + 4c_0 x^4}{2x}\lambda_3}{F^2_{\lambda_3}(\mathbf{t}_x)}\right). \tag{21}$$

Let us employ previous results to determine the action on $\mathfrak{sl}(2,\mathbb{R})$ of those elements g close to the neutral element $e \in SL(2,\mathbb{R})$. Using the so-called *canonical coordinates of the second kind* [52], we can write g within an open neighborhood U of e in a unique form as

$$g = \exp(-\lambda_3 \mathrm{a}_3)\exp(-\lambda_2 \mathrm{a}_2)\exp(-\lambda_1 \mathrm{a}_1), \tag{22}$$

for real constants λ_1, λ_2 and λ_3. This allows us to obtain the action of every $g \in U$ on $\mathrm{T}\mathbb{R}_0$ through the composition of the actions of elements $\exp(-\lambda_\alpha \mathrm{a}_\alpha)$, with $\lambda_\alpha \in \mathbb{R}$ for $\alpha = 1, 2, 3$. To do so, we determine the constants λ_1, λ_2 and λ_3 associated to each $g \in U$ in (22).

Considering the standard matrix representation of $SL(2,\mathbb{R})$, we can express every $g \in SL(2,\mathbb{R})$ as

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha\delta - \beta\gamma = 1, \qquad \alpha, \beta, \gamma, \delta \in \mathbb{R}. \tag{23}$$

In view of (17), and comparing (22) and (23), we obtain

$$\alpha = e^{\lambda_2/2}, \quad \beta = -e^{\lambda_2/2}\lambda_1, \qquad \gamma = e^{\lambda_2/2}\lambda_3.$$

Consequently,

$$\lambda_1 = -\beta/\alpha, \qquad \lambda_2 = 2\log\alpha, \qquad \lambda_3 = \gamma/\alpha,$$

and, from the basis (17), the decomposition (22) and expressions (18) and (21), the action reads

$$\varphi_{2KS}(g, \mathbf{t}_x) = \left(\frac{x}{F_g(\mathbf{t}_x)}, \frac{1}{F_g^2(\mathbf{t}_x)}\left[(v\alpha - 2x\beta)\left(\delta - \frac{\gamma v}{2x}\right) - 2c_0 x^3 \alpha\gamma\right]\right),$$

where

$$F_g(\mathbf{t}_x) = \left(\delta - \frac{\gamma v}{2x}\right)^2 + c_0 x^2 \gamma^2.$$

Although this expression has been derived for g being close to e, it can be proved that the action is properly defined at points $(g, \mathbf{t}_x)$ such that $F_g(\mathbf{t}_x) \neq 0$. If $c_0 > 0$, then $F_g(\mathbf{t}_x) > 0$

for all $g \in SL(2,\mathbb{R})$ and $\mathbf{t}_x \in \mathrm{T}\mathbb{R}_0$. So, φ_{2KS} becomes globally defined. Otherwise, $F_g(\mathbf{t}_x) > 0$ for g close enough to e. Then, φ_{2KS} is only defined on a neighborhood of e.

The action φ_{2KS} also permits us to write the general solution of system (13) in the form $(x(t), v(t)) = \varphi_{2KS}(g(t), \mathbf{t}_x)$, with $g(t)$ being a particular solution of

$$\frac{dg}{dt} = -Y_3^R(g) - b_1(t)Y_1^R(g), \tag{24}$$

where Y_α^R, with $\alpha = 1, 2, 3$, are the single right-invariant vector fields on $SL(2,\mathbb{R})$ such that $Y_\alpha^R(e) = \mathrm{a}_\alpha$ [11, 13]. Additionally, as $x(t)$ is the general solution of KS-2 equation (12), we readily see that

$$x(t) = \tau \circ \varphi_{2KS}(g(t), \mathbf{t}_x), \tag{25}$$

with $\tau : (x, v) \in \mathrm{T}\mathbb{R} \mapsto x \in \mathbb{R}$ a tangent bundle projection, provides us with the general solution of (12) in terms of a particular solution of (24).

Conversely, we prove that we can recover a particular solution to (24) from the knowledge of the general solution of (12). For simplicity, we will determine the particular solution $g_1(t)$ with $g_1(0) = e$. Given two particular solutions $x_1(t)$ and $x_2(t)$ of (12) with $dx_1/dt(t) = dx_2/dt(t) = 0$, the expression (25) implies that

$$(x_i(t), v_i(t)) = \varphi_{2KS}(g_1(t), (x_i(0), 0)), \qquad i = 1, 2.$$

Writing the above expression explicitly, we get

$$\begin{aligned} -\frac{x_i(0)v_i(t)}{2x_i^2(t)} &= \beta(t)\delta(t) + c_0 x_i^2(0)\alpha(t)\gamma(t), \\ \frac{x_i(0)}{x_i(t)} &= \delta^2(t) + c_0 x_i^2(0)\gamma^2(t), \end{aligned} \tag{26}$$

for $i = 1, 2$. The first two equations allow us to determine the value of $\beta(t)\delta(t)$ and $\alpha(t)\gamma(t)$. Meanwhile, we can obtain the value of $\delta^2(t)$ and $\gamma^2(t)$ from the other two ones. As $\delta(0) = 1$, we know that $\delta(t)$ is positive when close to $t = 0$. Taking into account that we have already worked out $\delta^2(t)$, we can determine $\delta(t)$ for small values of t. Since we have already obtained $\beta(t)\delta(t)$, we can also derive $\beta(t)$ for small values of t by using $\delta(t)$. Note that $\alpha(0) = 1$. So, $\alpha(t)$ is positive for small values of t, and the sign of $\alpha(t)\gamma(t)$ determines the sign of $\gamma(t)$ around $t = 0$. In view of this, the value of $\gamma(t)$ can be determined from $\gamma^2(t)$ in the interval around $t = 0$. Summing up, we can obtain algebraically a particular solution of (26) with $g_1(0) = e$ from the general solution of (12).

4 The third-order Kummer–Schwarz equations

The results obtained in the previous section can be generalized directly to the case of KS-3 equations, i.e., the third-order differential equations

$$\frac{d^3x}{dt^3} = \frac{3}{2}\left(\frac{dx}{dt}\right)^{-1}\left(\frac{d^2x}{dt^2}\right)^2 - 2c_0(x)\left(\frac{dx}{dt}\right)^3 + 2b_1(t)\frac{dx}{dt}, \tag{27}$$

where $c_0 = c_0(x)$ and $b_1 = b_1(t)$ are arbitrary.

The relevance of KS-3 equations resides in their relation to the Kummer's problem [27, 28, 30], Milne–Pinney [26] and Riccati equations [26, 37, 53]. Such relations can be useful in the interpretation of physical systems through KS-3 equations, e.g., the case of quantum non-equilibrium dynamics of many body systems [54]. Furthermore, KS-3 equations with $c_0 = 0$ can be rewritten as $\{x, t\} = 2b_1(t)$, where $\{x, t\}$ is the *Schwarzian derivative* [55] of the function $x(t)$ with respect to t.

Let us write KS-3 equations as a first-order system

$$\begin{cases} \dfrac{dx}{dt} = v, \\ \dfrac{dv}{dt} = a, \\ \dfrac{da}{dt} = \dfrac{3}{2}\dfrac{a^2}{v} - 2c_0(x)v^3 + 2b_1(t)v, \end{cases} \tag{28}$$

in the open submanifold $\mathcal{O}_2 = \{(x, v, a) \in \mathrm{T}^2\mathbb{R} \mid v \neq 0\}$ of $\mathrm{T}^2\mathbb{R} \simeq \mathbb{R}^3$, the referred to as *second-order tangent bundle* [56] of $\mathbb{R}$.

Consider now the set of vector fields on $\mathcal{O}_2$ given by

$$\begin{aligned} N_1 &= 2v\frac{\partial}{\partial a}, \\ N_2 &= v\frac{\partial}{\partial v} + 2a\frac{\partial}{\partial a}, \\ N_3 &= v\frac{\partial}{\partial x} + a\frac{\partial}{\partial v} + \left(\frac{3}{2}\frac{a^2}{v} - 2c_0(x)v^3\right)\frac{\partial}{\partial a}, \end{aligned} \tag{29}$$

which satisfy the commutation relations

$$[N_1, N_3] = 2N_2, \quad [N_1, N_2] = N_1, \quad [N_2, N_3] = N_3. \tag{30}$$

Thus, they span a three-dimensional Lie algebra of vector fields V isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Since (28) is determined by the t-dependent vector field

$$X_t = v\frac{\partial}{\partial x} + a\frac{\partial}{\partial v} + \left(\frac{3}{2}\frac{a^2}{v} - 2c_0(x)v^3 + 2b_1(t)v\right)\frac{\partial}{\partial a},$$

we can write $X_t = N_3 + b_1(t)N_1$. Consequently, X takes values in the finite-dimensional Vessiot–Guldberg Lie algebra V and becomes an $\mathfrak{sl}(2, \mathbb{R})$-Lie system. This generalizes the result provided in [45] for $c_0(x) = const.$

We shall now reduce the integration of (28) with $c_0(x) = const.$, and in consequence the integration of the related (27), to working out a particular solution of the Lie system (24). To do so, we employ the Lie group action $\varphi_{3KS} : SL(2, \mathbb{R}) \times \mathcal{O}_2 \to \mathcal{O}_2$ whose infinitesimal action is given by the Lie algebra isomorphism $\rho : \mathfrak{sl}(2, \mathbb{R}) \to V$ satisfying that $\rho(\mathrm{a}_\alpha) = N_\alpha$, with $\alpha = 1, 2, 3$. This Lie group action holds that

$$\frac{d}{ds}\varphi_{3KS}(\exp(-s\mathrm{a}_\alpha), \mathbf{t}^2_x) = N_\alpha(\varphi_{3KS}(\exp(-s\mathrm{a}_\alpha), \mathbf{t}^2_x)),$$

with $\mathbf{t}_x^2 \equiv (x, v, a) \in \mathcal{O}_2$ and $\alpha = 1, 2, 3$. Integrating N_1 and N_2, we easily see that

$$\varphi_{3KS}\left(\exp(-\lambda_1 \mathrm{a}_1), \mathbf{t}_x^2\right) = \begin{pmatrix} x \\ v \\ a + 2v\lambda_1 \end{pmatrix}$$

and

$$\varphi_{3KS}\left(\exp(-\lambda_2 \mathrm{a}_2), \mathbf{t}_x^2\right) = \begin{pmatrix} x \\ ve^{\lambda_2} \\ ae^{2\lambda_2} \end{pmatrix}.$$

To integrate N_3, we need to obtain the solutions of

$$\frac{dx}{d\lambda_3} = v, \qquad \frac{dv}{d\lambda_3} = a, \qquad \frac{da}{d\lambda_3} = \frac{3}{2}\frac{a^2}{v} - 2c_0 v^3. \tag{31}$$

Proceeding, *mutatis mutandis*, as in the analysis of system (19), we obtain

$$v(\lambda_3) = \frac{v(0)}{F_{\lambda_3}(x(0), v(0), a(0))},$$

with

$$F_{\lambda_3}(\mathbf{t}_x^2) = \left(1 - \frac{a\lambda_3}{2v}\right)^2 + c_0 v^2 \lambda_3^2.$$

Taking into account this and the first two equations within (31), we see that

$$\varphi_{3KS}\left(e^{-\lambda_3 \mathrm{a}_3}, \mathbf{t}_x^2\right) = \begin{pmatrix} x + v\int_0^{\lambda_3} F^{-1}_{\lambda_3'}(\mathbf{t}_x^2) d\lambda_3' \\ F^{-1}_{\lambda_3}(\mathbf{t}_x^2) v \\ v\partial(F^{-1}_{\lambda_3}(\mathbf{t}_x^2))/\partial\lambda_3 \end{pmatrix}.$$

Using decomposition (22), we can reconstruct the new action

$$\varphi_{3KS}\left(g, \mathbf{t}_x^2\right) = \begin{pmatrix} x + v\int_0^{\gamma/\alpha} \bar{F}^{-1}_{\lambda_3,g}(\mathbf{t}_x^2) d\lambda_3 \\ \bar{F}^{-1}_{\gamma/\alpha,g}(\mathbf{t}_x^2) v \\ v\frac{\partial(\bar{F}^{-1}_{\lambda_3,g}(\mathbf{t}_x^2))}{\partial\lambda_3}\Big|_{\lambda_3=\gamma/\alpha} \end{pmatrix},$$

with $\bar{F}_{\lambda_3,g}(\mathbf{t}_x^2) = \alpha^{-2} F_{\lambda_3}(x, v\alpha^2, (a\alpha - 2v\beta)\alpha^3\lambda_3)$, i.e.,

$$\bar{F}_{\lambda_3,g}(\mathbf{t}_x^2) = \left(\frac{1}{\alpha} - \frac{a\alpha - 2v\beta}{2v}\lambda_3\right)^2 + c_0 v^2 \alpha^2 \lambda_3^2.$$

This action enables us to write the general solution of (28) as

$$(x(t), v(t), a(t)) = \varphi_{3KS}(g(t), \mathbf{t}_x^2),$$

where $\mathbf{t}_x^2 \in \mathcal{O}_2$ and $g(t)$ is a particular solution of the equation on $SL(2,\mathbb{R})$ given by (24). Hence, if $\tau^{2)} : (x, v, a) \in \mathrm{T}^2\mathbb{R} \mapsto x \in \mathbb{R}$ is the fiber bundle projection corresponding to the second-order tangent bundle on $\mathbb{R}$, we can write the general solution of (27) in the form

$$x(t) = \tau^{2)} \circ \varphi_{3KS}(g(t), \mathbf{t}_x^2),$$

where $g(t)$ is any particular solution of (24).

Conversely, given the general solution of (27), we can obtain a particular solution of (24). As before, we focus on obtaining the particular solution $g_1(t)$, with $g_1(0) = e$. In this case, given two particular solution $x_1(t), x_2(t)$ of (27) with $d^2x_1/dt^2(0) = d^2x_2/dt^2(t) = 0$, we obtain that the t-dependent coefficients $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ corresponding to the matrix expression of $g_1(t)$ obey a system similar to (26) where v and x have been replaced by a and v, respectively.

5 On the relations of Kummer–Schwarz equations with other equations

We have already shown that the general solution of second- and third-order Kummer–Schwarz equations can be obtained from a particular solution of a Lie system in $SL(2,\mathbb{R})$ and vice versa. In this section, we will show that this property is shared by all systems that are known to be closely related to KS-2 and KS-3 equations: e.g., time-dependent frequency harmonic oscillators, Milne–Pinney and Riccati equations [25, 29, 30, 37, 51]. This allows us to explain why the integration of one of these systems amounts to integrating a particular instance of all the others. In addition, we find new remarkable systems of differential equations that are related in this same way to second- and third-order Kummer–Schwarz equations.

Let us start by analyzing the Riccati equations of the form

$$\frac{dx}{dt} = b_1(t) + x^2, \tag{32}$$

These equations are determined by a time-dependent vector field

$$W_t = (b_1(t) + x^2)\frac{\partial}{\partial x},$$

which can be written as $X_t = W_3 + b_1(t)W_1$, where

$$W_1 = \frac{\partial}{\partial x}, \qquad W_2 = x\frac{\partial}{\partial x}, \qquad W_3 = x^2\frac{\partial}{\partial x}$$

satisfy the commutating relations

$$[W_1, W_3] = 2W_2, \quad [W_1, W_2] = W_1, \quad [W_2, W_3] = W_3. \tag{33}$$

Hence, equations (32) are Lie systems related to a Vessiot–Guldberg Lie Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. This Lie algebra gives rise to a local Lie group action $\varphi : SL(2,\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ of the form

$$\varphi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, x\right) = \frac{\alpha x - \beta}{-\gamma x + \delta}, \qquad \alpha\delta - \beta\gamma = 1,$$

whose fundamental vector fields associated with $\mathrm{a}_1, \mathrm{a}_2$ and a_3 are W_1, W_2 and W_3, respectively. In consequence, the solution of W can be put in the form $x(t) = \varphi(g(t), x_0)$, with $x_0 \in \mathbb{R}$ and $g(t)$ given by (24). Conversely, given three different particular solutions $x_1(t)$, $x_2(t)$ and $x_3(t)$ of (32), we can easily determine $g(t)$, with $g(0) = e$, from the equations $x_i(t) = \varphi(g(t), x_i(0))$, with $i = 1, 2, 3$.

Consider now the Milne–Pinney equations

$$\frac{d^2x}{dt^2} = -b_1(t)x + \frac{c}{x^3},$$

with $c \in \mathbb{R}$ and $b_1(t)$ being an arbitrary t-dependent function. It is remarkable that when $c = 0$, we have a t-dependent frequency harmonic oscillator. The Milne–Pinney equations in form of a first-order system

$$\begin{cases} \dfrac{dx}{dt} = v, \\ \dfrac{dv}{dt} = -b_1(t)x + \dfrac{c}{x^3} \end{cases} \tag{34}$$

is governed by the time-dependent vector field $W = W_3 + b_1(t)W_1$, where

$$W_1 = -x\frac{\partial}{\partial v}, \qquad W_2 = \frac{1}{2}\left(v\frac{\partial}{\partial v} - x\frac{\partial}{\partial x}\right), \qquad W_3 = v\frac{\partial}{\partial x} + \frac{c}{x^3}\frac{\partial}{\partial v}. \tag{35}$$

For simplicity, we restrict ourselves to the case $x > 0$ and $c > 0$. The above vector fields close on a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and give rise to the Lie group action φ_{MP} : $(A, (x, v)) \in SL(2,\mathbb{R}) \times \mathrm{T}\mathbb{R}_+ \mapsto (\bar{x}, \bar{v}) \in \mathrm{T}\mathbb{R}_+$ given by

$$\bar{x} = \sqrt{\frac{c + [(\alpha v + \beta x)(\gamma v + \delta x) + c(\alpha\gamma/x^2)]^2}{(\alpha v + \beta x)^2 + c\alpha^2/x^2}},$$

$$\bar{v} = \kappa\sqrt{(\alpha v + \beta x)^2 + \frac{c\alpha^2}{x^2}\left(1 - \frac{x^2}{\alpha^2\bar{x}^2}\right)},$$

where

$$\kappa = \operatorname{sign}\left(\frac{c\alpha\gamma}{x^2} + (\alpha v + \beta x)(\gamma v + \delta x)\right).$$

This expression can be obtained proceeding as in previous sections or as in [51]. This shows that the general solution to (34) can be obtained through a particular solution to (24). Conversely, given two particular solutions $(x_1(t), v_1(t))$ and $(x_2(t), v_2(t))$ to (34), e.g. those with $v_1(0) = v_2(0) = 0$, a particular solution to (24) with $g(0) = e$ can be obtained by solving the algebraic system of equations $\varphi_{MP}(g(t), (x_i(0), v_i(0))) = (x_i(t), v_i(t))$, where $i = 1, 2$.

Let us analyze a last example of Lie system related to $\mathfrak{sl}(2,\mathbb{R})$. Consider

$$\begin{cases} \dfrac{dx}{dt} = b_1(t) + x^2, \\ \dfrac{dy}{dt} = 2x, \\ \dfrac{dz}{dt} = -e^y. \end{cases} \tag{36}$$

appearing in the application of the Wei-Norman method to $\mathfrak{sl}(2,\mathbb{R})$-Lie systems [13, 44]. We define

$$W_1 = \frac{\partial}{\partial x}, \quad W_2 = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad W_3 = x^2\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y} - e^y\frac{\partial}{\partial z},$$

which close on the commutation relations (33). In view of these vector fields, we easily see that system (36) is a Lie system governed by a time-dependent vector field $W_t = W_3 + b_1(t)W_1$.

The integration of W_1, W_2 and W_3 results in an action $\varphi_C : SL(2,\mathbb{R}) \times \mathbb{R}^3 \to \mathbb{R}^3$ of the form

$$\varphi_C\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} \frac{\alpha x - \beta}{-\gamma x + \delta} \\ y - \log\left(\delta - \gamma x\right)^2 \\ z - \frac{e^y \gamma}{\delta - \gamma x} \end{pmatrix},$$

which allows us to write the general solution of (36) as

$$(x(t), y(t), z(t)) = \varphi_C(g(t), (x_0, y_0, z_0)), \tag{37}$$

with $g(t)$ being a particular solution of (6) and $(x_0, y_0, z_0) \in \mathbb{R}^3$. Again, it is easy to prove that certain particular solutions to (36) give rise to a solution of (24) by solving the corresponding system induced by (37).

Note that our results show that solving second- and third-order Kummer–Schwarz equations, time-dependent frequency harmonic oscillators, Milne–Pinney equations, and Riccati equations amounts to obtaining a particular solution of (24). This provides a new geometric and unified explanation of the relations among the solutions of these systems presented in different forms, e.g., through specific non-local changes of variables, in the literature [26, 37]. Moreover, our results can be potentially be extended to any $\mathfrak{sl}(2,\mathbb{R})$-Lie system whose Vessiot–Guldberg Lie algebra can be integrated into an action [13, 57].

6 On the properties of the Schwarzian derivative

The Schwarzian derivative of a real function $f = f(t)$ is defined by

$$\{f, t\} = \frac{d^3 f}{dt^3}\left(\frac{df}{dt}\right)^{-1} - \frac{3}{2}\left[\frac{d^2 f}{dt^2}\left(\frac{df}{dt}\right)^{-1}\right]^2.$$

This derivative is clearly related to KS-3 equations (27) with $c_0 = 0$, which can be written as $\{f, t\} = 2b_1(t)$.

Although a superposition rule for studying KS-3 equations was developed in [24], the result provided in there was not valid when $c_0 = 0$, which retrieves the relevant equation $\{x, t\} = 2b_1(t)$. This is why we aim to reconsider this case and its important connection to the Schwarzian derivative.

We shall now follow the method detailed in the introduction to obtain a superposition rule for (28). The vector fields N_1, N_2, N_3 are linearly independent at a generic point of $\mathcal{O}_2 \subset \mathrm{T}^2\mathbb{R}_0$. Therefore, obtaining a superposition rule for (28) amounts to obtaining three functionally independent first-integrals common to all diagonal prolongations $\widetilde{N}_1, \widetilde{N}_2, \widetilde{N}_3$ in $(\mathcal{O}_2)^2$ satisfying (10). As $[\widetilde{N}_1, \widetilde{N}_3] = 2\widetilde{N}_2$, it suffices to obtain common first-integrals for $\widetilde{N}_1, \widetilde{N}_3$ to describe first-integrals common to the integrable distribution $\mathcal{D}$ spanned by $\widetilde{N}_1, \widetilde{N}_2, \widetilde{N}_3$.

Let us start by solving $\widetilde{N}_1 F = 0$, with $F : \mathcal{O}_2 \to \mathbb{R}$, i.e.,

$$v_0 \frac{\partial F}{\partial a_0} + v_1 \frac{\partial F}{\partial a_1} = 0.$$

The method of characteristics shows that F must be constant along the solutions of the associated *Lagrange–Charpit equations* [58], namely

$$\frac{da_0}{v_0} = \frac{da_1}{v_1}, \qquad dx_0 = dx_1 = dv_0 = dv_1 = 0.$$

Such solutions are the curves $(x_0(\lambda), v_0(\lambda), a_0(\lambda), x_1(\lambda), v_1(\lambda), a_1(\lambda))$ within $\mathcal{O}_2$ with $\Delta = v_1(\lambda)a_0(\lambda) - a_1(\lambda)v_0(\lambda)$, for a real constant $\Delta \in \mathbb{R}$, and constant $x_i(\lambda)$ and $v_i(\lambda)$, with $i = 0, 1$. In other words, there exists a function $F_2 : \mathbb{R}^5 \to \mathbb{R}$ such that $F(x_0, v_0, a_0, x_1, v_1, a_1) = F_2(\Delta, x_0, x_1, v_0, v_1)$.

If we now impose $\widetilde{N}_3 F = 0$, we obtain

$$\widetilde{N}_3 F = \widetilde{N}_3 F_2 = \frac{\Delta + a_1 v_0}{v_1} \frac{\partial F_2}{\partial v_0} + a_1 \frac{\partial F_2}{\partial v_1} + v_0 \frac{\partial F_2}{\partial x_0} + v_1 \frac{\partial F_2}{\partial x_1} + \frac{3\Delta^2 + 6\Delta a_1 v_0}{2 v_1 v_0} \frac{\partial F_2}{\partial \Delta} = 0.$$

We can then write that $\widetilde{N}_2 F_2 = (a_1/v_1)\Xi_1 F_2 + \Xi_2 F_2 = 0$, where

$$\Xi_1 = v_0 \frac{\partial}{\partial v_0} + v_1 \frac{\partial}{\partial v_1} + 3\Delta \frac{\partial}{\partial \Delta}, \qquad \Xi_2 = v_0 \frac{\partial}{\partial x_0} + v_1 \frac{\partial}{\partial x_1} + \frac{\Delta}{v_1} \frac{\partial}{\partial v_0} + \frac{3\Delta^2}{2 v_0 v_1} \frac{\partial}{\partial \Delta}.$$

As F_2 does not depend on a_1 in the chosen coordinate system, it follows $\Xi_1 F_2 = \Xi_2 F_2 = 0$. Using the characteristics method again, we obtain that $\Xi_1 F_2 = 0$ implies the existence of a new function $F_3 : \mathbb{R}^4 \to \mathbb{R}$ such that $F_2(\Delta, x_0, x_1, v_0, v_1) = F_3(K_1 \equiv v_1/v_0, K_2 \equiv v_0^3/\Delta, x_0, x_1)$.

The only condition remaining is $\Xi_2 F_3 = 0$. In the local coordinate system $\{K_1, K_2, x_0, x_1\}$, this equation reads

$$v_0 \left(\frac{3}{2K_1} \frac{\partial F_3}{\partial K_2} - \frac{1}{K_2} \frac{\partial F_3}{\partial K_1} + \frac{\partial F_3}{\partial x_0} + K_1 \frac{\partial F_3}{\partial x_1} \right) = 0,$$

and its Lagrange–Charpit equations becomes

$$-K_2 dK_1 = \frac{2K_1 dK_2}{3} = dx_0 = \frac{dx_1}{K_1}.$$

From the first equality, we obtain that $K_1^3K_2^2 = \Upsilon_1$ for a certain real constant Υ_1. In view of this and with the aid of the above system, it turns out

$$\frac{2}{3}K_1^2 dK_2 = dx_1 \longrightarrow \frac{2}{3}\Upsilon_1^{2/3}K_2^{-4/3}dK_2 = dx_1.$$

Integrating, we see that $-2K_2K_1^2 - x_1 = \Upsilon_2$ for a certain real constant Υ_2. Finally, these previous results are used to solve the last part of the Lagrange–Charpit system, i.e.,

$$dx_0 = \frac{dx_1}{K_1} = \frac{4\Upsilon_1 dx_1}{(x_1 + \Upsilon_2)^2} \longrightarrow \Upsilon_3 = x_0 + \frac{4\Upsilon_1}{x_1 + \Upsilon_2}.$$

Note that $\partial(\Upsilon_1, \Upsilon_2, \Upsilon_3)/\partial(x_0, v_0, a_0) \neq 0$. Therefore, considering $\Upsilon_1 = k_1$, $\Upsilon_2 = k_2$ and $\Upsilon_3 = k_3$, we can obtain a mixed superposition rule. From these equations, we easily obtain

$$x_0 = \frac{x_1k_3 + k_2k_3 - 4k_1}{x_1 + k_2}. \tag{38}$$

Multiplying numerator and denominator of the right-hand side by a non-null constant Υ_4, the above expression can be rewritten as

$$x_0 = \frac{\alpha x_1 + \beta}{\gamma x_1 + \delta}, \tag{39}$$

with $\alpha = \Upsilon_4 k_3, \beta = \Upsilon_4(k_2k_3 - 4k_1), \gamma = \Upsilon_4, \delta = k_2\Upsilon_4$. Observe that

$$\alpha\delta - \gamma\beta = \Upsilon_4^2\Upsilon_1 = \frac{\Upsilon_4^2 v_0^3 v_1^3}{(v_1a_0 - a_1v_0)^2} \neq 0.$$

Then, choosing an appropriate Υ_4, we obtain that (38) can be rewritten as (39) for a family of constants $\alpha, \beta, \gamma, \delta$ such that $\alpha\delta - \gamma\beta = \pm 1$. It is important to recall that the matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad I = \alpha\delta - \beta\gamma = \pm 1,$$

are the matrix description of the Lie group $PGL(2, \mathbb{R})$.

Operating, we also obtain that

$$v_0 = \frac{Iv_1}{(\gamma x_1 + \delta)}, \qquad a_0 = I\left[\frac{a_1}{(\gamma x_1 + \delta)^2} - \frac{2v_1^2\gamma}{(\gamma x_1 + \delta)^3}\right].$$

The above expression together with (39) become a superposition rule for KS-3 equations with $c_0 = 0$ (written as a first-order system). In other words, the general solution $(x(t), v(t), a(t))$ of (27) with $c_0 = 0$ can be written as

$$(x(t), v(t), a(t)) = \Phi(A, x_1(t), v_1(t), a_1(t)),$$

with $(x_1(t), v_1(t), a_1(t))$ being a particular solution, $A \in PGL(2,\mathbb{R})$ and

$$\Phi(A, x_1, v_1, a_1) = \left(\frac{\alpha x_1 + \beta}{\gamma x_1 + \delta}, \frac{I v_1}{(\gamma x_1 + \delta)^2}, I \left[\frac{a_1(\gamma x_1 + \delta) - 2v_1^2 \gamma}{(\gamma x_1 + \delta)^3} \right] \right).$$

Moreover, $x(t)$, which is the general solution of a KS-3 equation with $c_0 = 0$, can be determined out of a particular solution $x_1(t)$ and three constants through

$$x(t) = \tau^{2)} \circ \Phi\left(A, x_1(t), \frac{dx_1}{dt}(t), \frac{d^2 x_1}{dt^2}(t) \right), \tag{40}$$

where we see that the right-hand part does merely depend on A and $x_1(t)$. This constitutes a *basic superposition rule* [24] for equations $\{x(t), t\} = 2b_1(t)$, i.e., it is an expression that allows us to describe the general solution of any of these equations in terms of a particular solution (without involving its derivatives) and some constants to be related to initial conditions. We shall now employ this superposition rule to describe some properties of the Schwarzian derivative.

From the equation above, we analyze the relation between two particular solutions $x_1(t)$ and $x_2(t)$ of the same equation $\{x, t\} = 2b_1(t)$, i.e., $\{x_1(t), t\} = \{x_2(t), t\}$. Our basic superposition rule (40) tells us that from $x_1(t)$ we can generate every other solution of the equation. In particular, there must exist certain real constants c_1, c_2, c_3, c_4 such that

$$x_2(t) = \frac{c_1 x_1(t) + c_2}{c_3 x_1(t) + c_4}, \qquad c_1 c_4 - c_2 c_3 \neq 0.$$

In this way, we recover a relevant property of this type of equations [32].

Our basic superposition rule (40) also provides us with information about the Lie symmetries of $\{x(t), t\} = 2b_1(t)$. Indeed, note that (40) implies that the local Lie group action $\varphi : PGL(2,\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$

$$\varphi(A, x) = \frac{\alpha x + \beta}{\gamma x + \delta},$$

transforms solutions of $\{x(t), t\} = 2b_1(t)$ into solutions of the same equation. The prolongation [60, 61] $\widehat{\varphi} : PGL(2,\mathbb{R}) \times \mathrm{T}^2\mathbb{R}_0 \to \mathrm{T}^2\mathbb{R}_0$ of φ to $\mathrm{T}^2\mathbb{R}_0$, i.e.,

$$\widehat{\varphi}(A, \mathbf{t}^2_x) = \left(\frac{\alpha x + \beta}{\gamma x + \delta}, \frac{I v}{(\gamma x + \delta)^2}, I \frac{a(\gamma x + \delta) - 2\gamma v^2}{(\gamma x + \delta)^3} \right),$$

gives rise to a group of symmetries $\varphi(A, \cdot)$ of (28) when $c_0 = 0$. The fundamental vector fields of this action are spanned by

$$Z_1 = -\frac{\partial}{\partial x}, \qquad Z_2 = x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} + a\frac{\partial}{\partial a}, \qquad Z_3 = -\left(x^2 \frac{\partial}{\partial x} + 2vx\frac{\partial}{\partial v} + 2(ax + v^2)\frac{\partial}{\partial a} \right),$$

which close on a Lie algebra of vector fields isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and commute with X_t for every $t \in \mathbb{R}$. In addition, their projections onto $\mathbb{R}$ must be Lie symmetries of $\{x(t), t\} = 2b_1(t)$. Indeed, they read

$$S_1 = -\frac{\partial}{\partial x}, \qquad S_2 = x\frac{\partial}{\partial x}, \qquad S_3 = -x^2 \frac{\partial}{\partial x},$$

which are the known Lie symmetries for these equations [62].

Consider now the equation $\{x(t), t\} = 0$. Obviously, this equation admits the particular solution $x(t) = t$. This, together with our basic superposition rule, show that the general solution of this equation is

$$x(t) = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad \alpha\delta - \gamma\beta \neq 0, \tag{41}$$

recovering another relevant known solution of these equations.

7 The Kummer–Schwarz equations and mixed superposition rules

In previous sections, we have provided an alternative and unified approach for the study of relations between Kummer–Schwarz equations and other remarkable differential equations. In this section, we want to show that when we describe all these equations by a Lie system with a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, we can retrieve and generalise certain expressions found in the literature. With this aim, we shall now apply the fundamentals explained in Section 2 to derive a mixed superposition rule for KS-2 equations.

As both (29) and (35) close on the same commutation relations, we can build up the family of vector fields $\widehat{W}_\alpha = W_\alpha \times W_\alpha$, for $\alpha = 1, 2, 3$

$$\begin{aligned}
\widehat{X}_1 &= -x_1\frac{\partial}{\partial v_1} - x_2\frac{\partial}{\partial v_2},\\
\widehat{X}_2 &= \frac{1}{2}\left(v_1\frac{\partial}{\partial v_1} + v_2\frac{\partial}{\partial v_2} - x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2}\right),\\
\widehat{X}_3 &= v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2},
\end{aligned} \tag{42}$$

that are linearly independent at a generic point of $\mathbb{R}^4$. Consequently, the vector fields $\widetilde{X}_\alpha = \widehat{X}_\alpha \times M_\alpha$, with $\alpha = 1, 2, 3$, namely

$$\begin{aligned}
\widetilde{X}_1 &= -x_1\frac{\partial}{\partial v_1} - x_2\frac{\partial}{\partial v_2} + 2x\frac{\partial}{\partial v},\\
\widetilde{X}_2 &= x\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial v} + \frac{1}{2}\left(v_1\frac{\partial}{\partial v_1} + v_2\frac{\partial}{\partial v_2} - x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2}\right),\\
\widetilde{X}_3 &= v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2} + v\frac{\partial}{\partial x} + \left(\frac{3}{2}\frac{v^2}{x} - 2c_0x^3\right)\frac{\partial}{\partial v},
\end{aligned} \tag{43}$$

are also linearly independent at a generic point of $\mathbb{R}^6$. Besides, a mixed superposition rule for (13) can be obtained by determining a family of two functionally independent functions F_1 and F_2 playing the rôle of common first-integrals of (43) and such that $\partial(F_1, F_2)/\partial(x, v) \neq 0$.

As $[\widetilde{X}_1, \widetilde{X}_3] = 2\widetilde{X}_2$, we only need to derive first-integrals common to $\widetilde{X}_1$ and $\widetilde{X}_3$. Let us assume that $F : \mathbb{R}^6 \to \mathbb{R}$ is such a first-integral. Then, the equation $\widetilde{X}_1 F = 0$ reads

$$2x\frac{\partial F}{\partial v} - x_1\frac{\partial F}{\partial v_1} - x_2\frac{\partial F}{\partial v_2} = 0.$$

Their corresponding Lagrange–Charpit equations show that F must be constant along the solutions of the system

$$dx = dx_1 = dx_2 = 0, \qquad \frac{dv}{2x} = \frac{dv_1}{-x_1} = \frac{dv_2}{-x_2}.$$

In other words, F is constant along curves with constant x, x_1, x_2 and such that $x_1 v + 2xv_1 = \xi_1$ and $x_2 v + 2xv_2 = \xi_2$, for constants $\xi_1, \xi_2 \in \mathbb{R}$. Consequently, $F(x, x_1, x_2, v, v_1, v_2) = F_2(x, x_1, x_2, \xi_1, \xi_2)$ for a function $F_2 : \mathbb{R}^5 \to \mathbb{R}$. Using the coordinate system $\{x, x_1, x_2, \xi_1, \xi_2, v\}$, we obtain that

$$\widetilde{X}_3 F = \widetilde{X}_3 F_2 = v\left[\frac{\partial F_2}{\partial x} + \frac{3\xi_1}{2x}\frac{\partial F_2}{\partial \xi_1} + \frac{3\xi_2}{2x}\frac{\partial F_2}{\partial \xi_2}\right] + \sum_{i=1,2}\left(\frac{\xi_i - x_i v}{2x}\frac{\partial F_2}{\partial x_i} - 2c_0 x_i x^3\frac{\partial F_2}{\partial \xi_i}\right) = 0.$$

The previous expression has to be satisfied for all v, and F_2 does not depend on it. Taking into account that $\widetilde{X}_3 F = \Theta_0 F + v\Theta_1 F = 0$, where

$$\begin{aligned}\Theta_0 &= \frac{\partial}{\partial x} + \frac{3\xi_1}{2x}\frac{\partial F_2}{\partial \xi_1} - \frac{x_1}{2x}\frac{\partial}{\partial x_1} + \frac{3\xi_2}{2x}\frac{\partial}{\partial \xi_2} - \frac{x_2}{2x}\frac{\partial}{\partial x_2},\\ \Theta_1 &= \frac{\xi_1}{2x}\frac{\partial}{\partial x_1} + \frac{\xi_2}{2x}\frac{\partial}{\partial x_2} - 2c_0 x_1 x^3\frac{\partial}{\partial \xi_1} - 2c_0 x_2 x^3\frac{\partial}{\partial \xi_2},\end{aligned}$$

we obtain that $\Theta_0 F_2 = \Theta_1 F_2 = 0$. From the first equation, we see that

$$\frac{dx}{2x} = \frac{d\xi_1}{3\xi_1} = \frac{d\xi_2}{3\xi_2} = \frac{dx_1}{-x_1} = \frac{dx_2}{-x_2}.$$

Proceeding as above, we obtain that there must exist a function $F_3 : \mathbb{R}^4 \to \mathbb{R}$ such that $F_2(x, x_1, x_2, \xi_1, \xi_2) = F_3(\Gamma_1 \equiv xx_1^2, \Gamma_2 \equiv xx_2^2, \Xi_1 \equiv \xi_1^2/x^3, \Xi_2 \equiv \xi_2^2/x^3)$.

Finally, using the coordinate system $\{\Gamma_1, \Gamma_2, \Xi_1, \Xi_2\}$, we obtain that $\Theta_1 F_2 = \Theta_1 F_3 = 0$ reads

$$\sum_{i=1,2}\left(-4c_0\sqrt{\Xi_i\Gamma_i}\frac{\partial}{\partial \Xi_i} + \sqrt{\Xi_i\Gamma_i}\frac{\partial}{\partial \Gamma_i}\right) = 0,$$

which gives us two common first-integrals to $\widetilde{X}_1$ and $\widetilde{X}_3$, i.e.,

$$F_i \equiv \Xi_i + 4c_0\Gamma_i = \frac{(x_i v + 2xv_i)^2}{x^3} + 4c_0 xx_i^2, \quad i = 1, 2.$$

Note that $\partial(F_1, F_2)/\partial(x, v) \neq 0$. Then, we can derive a superposition rule from the equations $F_1 = I_1$ and $F_2 = I_2$ for $I_1, I_2 \in \mathbb{R}$. Using $F_1 = I_1$, we obtain

$$v = x\frac{\pm\sqrt{(I_1 - 4c_0 xx_1^2)x} - 2v_1}{x_1}, \tag{44}$$

and from $F_2 = I_2$, we reach to

$$x = 4W^2 \left[I_2 x_1^2 + I_1 x_2^2 \pm 2x_1 x_2 \sqrt{I_1 I_2 - 4^2 c_0 W^2} \right]^{-1}, \tag{45}$$

where $W = x_1 v_2 - v_1 x_2$ and $I_1 I_2 - 4^2 c_0 W^2 > 0$. Plugging the above expression into (44), we see that

$$v = -8W^2 \frac{I_2 x_1 v_1 + I_1 x_2 v_2 \pm (x_1 v_2 + v_1 x_2)[I_1 I_2 - 4^2 c_0 W^2]^{\frac{1}{2}}}{\left[I_2 x_1^2 + I_1 x_2^2 \pm 2x_1 x_2 \sqrt{I_1 I_2 - 4^2 c_0 W^2} \right]^2}. \tag{46}$$

These two expressions permit us to write the general solution $(x(t), v(t))$ of (13), i.e. a KS-2 equation written as a first-order system, in terms of two generic solutions $(x_1(t), v_1(t))$ and $(x_2(t), v_2(t))$ of (34), i.e., a time-dependent frequency harmonic oscillator in first-order form, as

$$(x(t), v(t)) = \Phi(x_1(t), v_1(t), x_2(t), v_2(t), I_1, I_2),$$

where the components of $\Phi = (\Phi_x, \Phi_v)$ are given by (45) and (46), respectively.

Note also that, given two solutions $(x_1(t), v_1(t))$ and $(x_2(t), v_2(t))$ of (34), the expression $W = x_1(t)v_2(t) - v_1(t)x_2(t)$ is a constant of motion. Then, we can redefine the constants as $k_2 = I_1/4W^2$ and $k_1 = I_2/4W^2$ to obtain an equivalent superposition rule

$$x = \left[k_1 x_1^2 + k_2 x_2^2 \pm 2x_1 x_2 \sqrt{k_1 k_2 - c_0 W^{-2}} \right]^{-1},$$

$$v = -2 \frac{k_1 x_1 v_1 + k_2 x_2 v_2 \pm (x_1 v_2 + v_1 x_2) \sqrt{k_1 k_2 - c_0 W^{-2}}}{\left[k_1 x_1^2 + K_2 x_2^2 \pm 2x_1 x_2 \sqrt{k_1 k_2 - c_0 W^{-2}} \right]^2}.$$

It is straightforward that the first expression above allows us to describe the general solution of KS-2 equations by means of two generic solutions of t-dependent frequency harmonic oscillators and their first-order derivatives. This result generalizes the expression obtained by Berkovich [30] for equations with $b_1(t) = const.$, which is only valid for two particular solutions of the harmonic oscillator with Wronskian equal to one.

8 Conclusions and Outlook

We have provided a new geometric approach to Kummer–Schwarz equations that retrieves previous results in a simpler and unified way. Among our future endeavors, we shall lay some attention upon the structure of Lie symmetries and develop discrete methods for the study of Lie systems, which could be of great use in the study of Kummer-Schwarz equations as well.

To conclude, we are very interested in developing the theory of *partial superposition rules* [12], as much as we are in the generalization of other expressions describing general solutions of Kummer–Schwarz equations [30] in terms of certain specific sets of particular solutions.

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