Lie–Hamilton systems: theory and applications

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Abstract

This work concerns the definition and analysis of a new class of Lie systems on Poisson manifolds enjoying rich geometric features: the Lie–Hamilton systems. We devise methods to study their superposition rules, time independent constants of motion and Lie symmetries, linearisability conditions, etc. Our results are illustrated by examples of physical and mathematical interest.

1 Introduction

The use of geometric tools for studying differential equations has been proved to be a very successful approach, as witnessed by the many works devoted to this topic over the years [2, 5, 11, 47, 55]. Among these methods, we here focus on the theory of Lie systems [16, 17, 39, 45, 56].

Lie systems form a family of systems of first-order ordinary differential equations whose general solutions can be written in terms of finite families of particular solutions and a set of constants by a particular type of functions, the so-called *superposition rules* [16, 17, 45, 56]. Moreover, Lie systems enjoy many geometrical properties [16, 17, 18, 33, 39, 56].

In modern geometric terms, the *Lie–Scheffers Theorem* [17] states that a Lie system amounts to a t-dependent vector field taking values in a finite-dimensional Lie algebra of vector fields, the so-called *Vessiot–Guldberg Lie algebra* [23, 40, 41, 51]. This condition is so restrictive that only few differential equations can be considered as Lie systems. Nevertheless, Lie systems appear in very important physical and mathematical problems [4, 21, 22, 23, 24, 29, 31, 32, 40, 41, 50, 51], which strongly motivates their analysis.

The first aim of this work is to uncover an interesting geometric feature shared by several Lie systems. More specifically, we show that second-order Riccati equations [27, 30, 34, 35, 37], second-order Kummer–Schwarz equations, as well as Smorodinsky–Winternitz oscillators (among other other remarkable examples) can be described by Lie systems associated to Vessiot–Guldberg Lie algebras of Hamiltonian vector fields (with respect to a certain Poisson structure). In this way, we highlight that this property deserves a thorough study, which is the main aim of the paper.

Previous examples suggest us the definition and analysis of a new type of Lie systems, the hereafter called *Lie–Hamilton systems*, admitting a plethora of geometric properties. For instance, their dynamics is governed by curves in finite-dimensional Lie algebras of functions (with respect to a Poisson structure). These geometrical objects, the hereafter named *Lie–Hamiltonian structures*, are a key to understand the characteristics of Lie–Hamilton systems.

Our achievements are employed to study superposition rules, Lie symmetries, constants of motion, and other features of Lie–Hamilton systems. It is noticeable that Lie–Hamiltonian structures allow us to use Poisson and symplectic geometric techniques to study Lie–Hamilton systems. Among other achievements, we prove that t-independent constants of motion of Lie–Hamilton systems form a *function group* [54], provide conditions for simultaneous linearisation of Lie–Hamilton systems and their related Poisson bivectors, and we describe properties of Lie symmetries of Lie–Hamilton systems.

All our achievements are exemplified by the analysis of Lie systems of physical and mathematical relevance. Furthermore, several new concepts related to t-dependent vector fields are introduced and briefly investigated as a tool to investigate Lie–Hamilton systems.

The structure of the paper goes as follows. Section 2 concerns the description of the notions and conventions about Poisson geometry and Lie algebras to be used throughout our paper. Section 3 is devoted to some concepts of the theory of t-dependent vector fields and Lie systems. In Section 4 the analysis of several remarkable Lie systems on Poisson manifolds leads us to introduce the concept of a Lie–Hamilton system, which encompasses such systems as particular cases. Subsequently, the Lie–Hamiltonian structures are introduced and analysed in Section 5. Next, we investigate several geometric properties of Lie–Hamilton in Section 6. Finally, Section 7 summarises our main results and present an outlook of our future research on these systems.

2 Fundamentals

For simplicity, we hereafter assume all mathematical objects to be real, smooth, and globally defined. This permits us to omit several minor technical problems so as to highlight the main aspects of our results.

Let us denote Lie algebras by pairs $(V, [\cdot, \cdot])$, where V stands for a real linear space endowed with a Lie bracket $[\cdot\,,\cdot] : V \times V \to V$. Given two subsets $\mathcal{A}, \mathcal{B} \subset V$, we write $[\mathcal{A}, \mathcal{B}]$ for the real linear space spanned by the Lie brackets between elements of $\mathcal{A}$ and $\mathcal{B}$, and we define $\mathrm{Lie}(\mathcal{B}, V, [\cdot, \cdot])$ to be the smallest Lie subalgebra of V containing $\mathcal{B}$. Note that $\mathrm{Lie}(\mathcal{B}, V, [\cdot, \cdot])$ is a well-defined object (it exists and is unique), which is generated by the elements of

$$\mathcal{B}, [\mathcal{B}, \mathcal{B}], [\mathcal{B}, [\mathcal{B}, \mathcal{B}]], [\mathcal{B}, [\mathcal{B}, [\mathcal{B}, \mathcal{B}]]], [[\mathcal{B}, \mathcal{B}], [\mathcal{B}, \mathcal{B}]], \ldots \tag{2.1}$$

From now on, we use $\mathrm{Lie}(\mathcal{B})$ and V to represent $\mathrm{Lie}(\mathcal{B}, V, [\cdot, \cdot])$ and $(V, [\cdot, \cdot])$, correspondingly, when their meaning is clear from context.

Given a fibre vector bundle $\mathrm{pr} : P \to N$, we denote by $\Gamma(\mathrm{pr})$ the $C^\infty(N)$-module of its smooth sections. So, if $\tau_N : TN \to N$ and $\pi_N : T^*N \to N$ are the canonical projections associated with the tangent and cotangent bundle to N, respectively, then $\Gamma(\tau_N)$ and $\Gamma(\pi_N)$ designate the $C^\infty(N)$-modules of vector fields and one-forms on $\mathbb{R}^n$, correspondingly.

We call *generalised distribution* $\mathcal{D}$ on a differentiable manifold N a function that sends each $x \in N$ to a linear subspace $\mathcal{D}_x \subset T_xN$. A generalised distribution is said to be *regular at* $x' \in N$ when the function $\mathrm{r} : N \to \mathbb{N} \cup \{0\}$ of the form $\mathrm{r} : x \in N \mapsto \dim \mathcal{D}_x \in \mathbb{N} \cup \{0\}$ is locally constant around x'. Similarly, $\mathcal{D}$ is regular on an open $U \subset N$ when r is constant on U. Finally, a vector field $Y \in \Gamma(\tau_N)$ is said to take values in $\mathcal{D}$, in short $Y \in \mathcal{D}$, when $Y_x \in \mathcal{D}_x$ for all $x \in N$. Likewise, similar notions can be defined for a *generalised codistribution*, namely a mapping relating every $x \in N$ to a linear subspace of T^*_xN.

In what follows, a *Poisson algebra* $(\mathfrak{W}, \star, \{\cdot, \cdot\})$ is a triple consisting of an $\mathbb{R}$-vectorial space $\mathfrak{W}$ and two bilinear maps on $\mathfrak{W}$, namely $\star$ and $\{\cdot, \cdot\}$, such that $\mathfrak{W}$ endowed with $\star$ becomes a commutative and associative $\mathbb{R}$-algebra and $(\mathfrak{W}, \{\cdot, \cdot\})$ is a Lie algebra whose Lie bracket, the so-called *Poisson bracket* of the Poisson algebra, satisfies the *Leibnitz rule* relative to $\star$, namely

$$\{f \star g, h\} = f \star \{g, h\} + \{f, h\} \star g, \qquad \forall f, g, h \in \mathfrak{W}.$$

In other words, $\{\cdot, h\}$ is a derivation of the $\mathbb{R}$-algebra $\mathfrak{W}$ for each $h \in \mathfrak{W}$.

A *Poisson manifold* is a pair $(N, \{\cdot, \cdot\})$ such that $(C^\infty(N), \cdot, \{\cdot, \cdot\})$ becomes a Poisson algebra with respect to the standard product "$\cdot$" of functions on N. The map $\{\cdot, \cdot\}$ is called the *Poisson structure* of the Poisson manifold. Observe that a Poisson structure is a derivation in each entry, which, as shown next, has relevant consequences.

On one hand, given an $f \in C^\infty(N)$, there exists a single vector field X_f on N, the referred to as *Hamiltonian vector field* associated with f, such that $X_f g = \{g, f\}$ for all $g \in C^\infty(N)$. The Jacobi identity for the Poisson structure therefore entails

$$X_{\{f,g\}} = -[X_f, X_g], \qquad \forall f, g \in C^\infty(N).$$

In other words, the mapping $f \mapsto X_f$ is a Lie algebra anti-homomorphism between the Lie algebras $(C^\infty(N), \{\cdot,\cdot\})$ and $(\Gamma(\tau_N), [\cdot,\cdot])$.

On the other hand, a Poisson structure determines a unique bivector field $\Lambda \in \Gamma(\bigwedge^2 TN)$ such that

$$\{f, g\} = \Lambda(df, dg), \qquad \forall f, g \in C^\infty(N). \tag{2.2}$$

We call Λ the *Poisson bivector* of the Poisson manifold $(N, \{\cdot,\cdot\})$. In view of the Jacobi identity for the Poisson structure, it follows that $[\Lambda, \Lambda]_S = 0$, with $[\cdot,\cdot]_S$ being the *Schouten–Nijenhuis Lie bracket* [52]. Conversely, every bivector field Λ on N satisfying the previous Schouten–Nijenhuis Lie bracket vanishing condition gives rise to a Poisson structure given by (2.2). This justifies referring to Poisson manifolds as $(N, \{\cdot,\cdot\})$ or (N, Λ) indistinctly. In some cases, we shall write $\{\cdot,\cdot\}_\Lambda$ for the Poisson structure induced by a Poisson bivector Λ if this is may not be clear from context.

Every Poisson bivector induces a unique bundle morphism $\widehat{\Lambda} : T^*N \to TN$ such that $\omega'(\widehat{\Lambda}(\omega)) = \Lambda(\omega, \omega')$ for every $\omega, \omega' \in \Gamma(\pi_N)$. This morphism allows us to relate every function $f \in C^\infty(N)$ to its associated vector field X_f through the relation $X_f = -\widehat{\Lambda}(df)$. We define $\mathrm{Ham}(N, \Lambda)$ to be the $\mathbb{R}$-linear space of Hamiltonian vector fields on N relative to Λ. This space induces an integrable generalised distribution $\mathcal{F}^\Lambda$ on N, the so-called *characteristic distribution* associated to Λ, of the form $\mathcal{F}^\Lambda_x = \{X_x \mid X \in \mathrm{Im}\,\widehat{\Lambda}\} \subset T_xN$, with $x \in N$, whose leaves are symplectic manifolds with respect to the restrictions of Λ [54].

If $X_f = 0$, we say that f is a *Casimir function*. We denote by $\mathrm{Cas}(N, \Lambda)$ the $\mathbb{R}$-linear space of Casimir functions on N relative to the Poisson bivector Λ. Finally, let us define a last structure that will be of interest in our work.

Definition 2.1. We call *Casimir co-distribution* of the Poisson manifold (N, Λ) the generalised co-distribution of the form $\mathcal{C}^\Lambda = \ker \widehat{\Lambda}$.

It is well known that the cotangent bundle of a Poisson manifold (N, Λ) admits a Lie algebroid structure $(T^*N, [\cdot,\cdot]_\Lambda, \widehat{\Lambda})$, with anchor $\widehat{\Lambda}$ and Lie bracket $[\omega, \omega']_\Lambda = \mathcal{L}_{\widehat{\Lambda}(\omega)}\omega' - \mathcal{L}_{\widehat{\Lambda}(\omega')}\omega - d\Lambda(\omega, \omega')$, where $\mathcal{L}_X$ denotes the Lie derivative with respect to a vector field X. In particular, $[df, dg]_\Lambda = d\{f, g\}$, for all $f, g \in C^\infty(N)$ (see [46, 53] for further details).

Proposition 2.2. *The Casimir co-distribution of a Poisson manifold is involutive.*

Proof. Given two sections $\omega, \omega' \in \mathcal{C}^\Lambda$, we have that $\widehat{\Lambda}(\omega) = \widehat{\Lambda}(\omega') = 0$ and then $\Lambda(\omega, \omega') = 0$. In consequence,

$$[\omega, \omega']_\Lambda = \mathcal{L}_{\widehat{\Lambda}(\omega)}\omega' - \mathcal{L}_{\widehat{\Lambda}(\omega')}\omega - d\Lambda(\omega, \omega') = 0.$$

□

3 Time-dependent vector fields and Lie systems

A *t-dependent vector field* on N is a map $X : (t, x) \in \mathbb{R} \times N \mapsto X(t, x) \in TN$ such that $\tau_N \circ X = \pi_2$, where $\pi_2 : (t, x) \in \mathbb{R} \times N \mapsto x \in N$. This condition entails that every t-dependent vector field amounts to a family of vector fields $\{X_t\}_{t \in \mathbb{R}}$, with $X_t : x \in N \mapsto X(t, x) \in TN$ for all $t \in \mathbb{R}$ and vice versa [23].

We call *integral curves* of X the integral curves $\gamma : \mathbb{R} \mapsto \mathbb{R} \times N$ of the *suspension* of X, i.e. the vector field $X(t, x) + \partial/\partial t$ on $\mathbb{R} \times N$ [1]. Every integral curve γ admits a parametrization in terms of a parameter $\bar{t}$ such that

$$\frac{d(\pi_2 \circ \gamma)}{d\bar{t}}(\bar{t}) = (X \circ \gamma)(\bar{t}).$$

This system is referred to as the *associated system* of X. Conversely, every system of first-order differential equations in normal form describes the integral curves of a unique t-dependent vector field. This establishes a bijection between t-dependent vector fields and systems of first-order differential equations in normal form, which justifies to use X to denote both a t-dependent vector field and its associated system.

Definition 3.1. The *minimal Lie algebra* of a t-dependent vector field X on N is the smallest real Lie algebra, V^X, containing the vector fields $\{X_t\}_{t\in\mathbb{R}}$, namely $V^X = \mathrm{Lie}(\{X_t\}_{t\in\mathbb{R}})$.

Minimal Lie algebras enable us to define the following new geometric structures that will be of interest so as to study the geometric properties of Lie systems, in general, and Lie–Hamilton systems, in particular.

Definition 3.2. Given a t-dependent vector field X on N, its *associated distribution*, $\mathcal{D}^X$, is the generalised distribution on N spanned by the vector fields of V^X, i.e.

$$\mathcal{D}^X_x = \{Y_x \mid Y \in V^X\} \subset T_xN,$$

and its *associated co-distribution*, $\mathcal{V}^X$, is the generalised co-distribution on N of the form

$$\mathcal{V}^X_x = \{\vartheta \in T^*_xN \mid \vartheta(Z_x) = 0, \forall\ Z_x \in \mathcal{D}^X_x\} = (\mathcal{D}^X_x)^\circ \subset T^*_xN,$$

where $(\mathcal{D}^X_x)^\circ$ is the *annihilator* of $\mathcal{D}^X_x$.

Observe that the function $\mathrm{r}^X : x \in N \mapsto \dim \mathcal{D}^X_x \in \mathbb{N}\cup\{0\}$ needs not be constant on N. We can only guarantee that $\mathrm{r}^X(x) = k$ implies $\mathrm{r}^X(x') \geq \mathrm{r}^X(x)$ for x' in a neighbourhood of x. Indeed, in this case there exist k vector fields $Y_1, \ldots, Y_k \in V^X$ such that $(Y_1)_x, \ldots, (Y_k)_x \in T_xN$ are linearly independent. As we assume vector fields to be smooth, $(Y_1)_{x'}, \ldots, (Y_k)_{x'} \in T_{x'}N$ are also linearly independent for x' in a neighbourhood of x and hence $\mathrm{r}^X(x') \geq k$. From here, it easily follows that r^X is a *lower semicontinuous function* and must be constant on the connected components of an open and dense subset U^X of N (cf. [52, p. 19]), where $\mathcal{D}^X$ becomes a regular distribution. As for every $x \in U^X$ there exists a local basis for $\mathcal{D}^X$ consisting of $\mathrm{r}^X(x)$ elements belonging to V^X, the generalised distribution $\mathcal{D}^X$ is involutive and integrable on each connected component of U^X. Since $\dim \mathcal{V}^X_x = \dim\, N - \mathrm{r}^X(x)$, then $\mathcal{V}^X$ becomes a regular co-distribution on each component also.

The most relevant instance for us is when $\mathcal{D}^X$ is determined by a finite-dimensional V^X and hence $\mathcal{D}^X$ becomes integrable on the whole N [48, p. 63]. It is worth noting that even in this case, $\mathcal{V}^X$ does not need to be a *differentiable distribution*, i.e. given $\vartheta \in \mathcal{V}^X_x$, it does not generally exist a locally defined one-form $\omega \in \mathcal{V}^X$ such that $\omega_x = \vartheta$.

Let us describe a first result that justifies the definition of the above geometric notions.

Proposition 3.3. *A function $f : U \to \mathbb{R}$ is a local t-independent constant of motion for a system X if and only if $df \in \mathcal{V}^X|_U$.*

Proof. Under the above assumptions, $X_tf|_U = df(X_t)|_U = 0$ for all $t \in \mathbb{R}$. Consequently, df also vanishes on the successive Lie brackets of elements from $\{X_t\}_{t\in\mathbb{R}}$ and hence

$$df(Y)|_U = Yf|_U = 0, \qquad \forall Y \in \mathrm{Lie}(\{X_t\}_{t\in\mathbb{R}}).$$

Since the elements of V^X span the generalised distribution $\mathcal{D}^X$, then $df_x(Z_x) = 0$ for all $x \in U$ and $Z_x \in \mathcal{D}^X_x$, i.e. $df \in \mathcal{V}^X|_U$. The converse directly follows from the above considerations. □

In brief, Proposition 3.3 shows that (locally defined) t-independent constants of motion of t-dependent vector fields are determined by (locally defined) exact one-forms taking values in its associated co-distribution. Then, $\mathcal{V}^X$ is what really matters in the calculation of such constants of motion for a system X.

Let us enunciate a lemma that will be used throughout our work and whose proof is straightforward.

Lemma 3.4. *Given a system X, its associated co-distribution $\mathcal{V}^X$ admits a local basis around every $x \in U^X$ of the form $df_1, \ldots, df_{p(x)}$, with $p(x) = \mathrm{r}^X(x)$ and $f_1, \ldots, f_{p(x)} : U \subset U^X \to \mathbb{R}$ being a family of (local) t-independent constants of motion for X. Furthermore, the $\mathbb{R}$-linear space $\mathcal{I}^X|_U$ of t-independent constants of motion of X on U can be written as*

$$\mathcal{I}^X|_U = \{g \in C^\infty(U) \mid \exists F : U \subset \mathbb{R}^{p(x)} \to \mathbb{R},\ g = F(f_1, \ldots, f_{p(x)})\}.$$

Note 3.5. Roughly speaking, the above lemma shows that $\mathcal{V}^X$ is differentiable on U^X.

Let us now turn to some fundamental notions appearing in the theory of Lie systems.

Definition 3.6. A *superposition rule* depending on m particular solutions for a system X on N is a function $\Phi : N^m \times N \to N$, $x = \Phi(x_{(1)}, \ldots, x_{(m)}; \lambda)$, such that the general solution $x(t)$ of X can be brought into the form $x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t); \lambda)$, where $x_{(1)}(t), \ldots, x_{(m)}(t)$ is any generic family of particular solutions and λ is a point of N to be related to initial conditions.

The conditions ensuring that a system X possesses a superposition rule are stated by the *Lie–Scheffers Theorem* [45, Theorem 44]. A modern statement of this relevant result is described next (for a modern geometric description see [17, Theorem 1]).

Theorem 3.7. *A system X admits a superposition rule if and only if X can be written as $X_t = \sum_{\alpha=1}^r b_\alpha(t) X_\alpha$ for a certain family $b_1(t), \ldots, b_r(t)$ of t-dependent functions and a collection $X_1, \ldots, X_r$ of vector fields spanning an r-dimensional real Lie algebra.*

Systems of first-order differential equations possessing a superposition rule are called *Lie systems*. The Lie–Scheffers Theorem yields that every Lie system X is related to (at least) one finite-dimensional real Lie algebra of vector fields V, the so-called *Vessiot–Guldberg Lie algebra*, satisfying that $\{X_t\}_{t\in\mathbb{R}} \subset V$. This implies that V^X must be finite-dimensional. Conversely, if V^X is finite-dimensional, this Lie algebra can be chosen as a Vessiot–Guldberg Lie algebra for X. This proves the following theorem, which motivates, among other reasons, the definition of V^X [23].

Theorem 3.8. (The abbreviated Lie–Scheffers Theorem) *A system X admits a superposition rule if and only if V^X is finite-dimensional.*

The Lie–Scheffers Theorem may be used to reduce the integration of a Lie system to solving a special type of Lie systems on a Lie group. More precisely, every Lie system X on a manifold N possessing a Vessiot–Guldberg Lie algebra V, let us say $X_t = \sum_{\alpha=1}^r b_\alpha(t) X_\alpha$, where $X_1 \ldots, X_r$ is a basis of V, can be associated with a (generally local) Lie group action $\varphi : G \times N \to N$ whose fundamental vector fields coincide with those of V [49, Theorem XI]. This action allows us to bring the general solution $x(t)$ of X into the form $x(t) = \varphi(g(t), x_0)$, where $x_0 \in N$ and $g(t)$ is the solution with $g(0) = e$ of the Lie system

$$\frac{dg}{dt} = -\sum_{\alpha=1}^r b_\alpha(t) X_\alpha^R(g), \qquad g \in G, \tag{3.1}$$

where $X_1^R, \ldots, X_r^R$ are a family of right-invariant vector fields on G admitting the same structure constants as $-X_1, \ldots, -X_r$ (see [16] for details). In this way, the explicit integration of a Lie system X reduces to finding one particular solution of (3.1) if φ is explicitly known. Conversely, the general solution of X enables us to construct the solution for (3.1) with $g(0) = e$ by solving an algebraic system of equations, provided the explicit form of φ is given [3].

4 Lie–Hamilton systems

A few instances of Lie systems on Poisson manifolds have recently appeared during the analysis of various mathematical and physical problems [4, 16, 26, 32]. In all these cases, and several new ones to be presented here, the structure of the Lie system can be related to the Poisson manifold in a special way. Let us analyse this question in depth to motivate our definition of a Lie–Hamilton system.

Consider a second-order Riccati equation, i.e. a second-order differential equation of the form

$$\frac{d^2x}{dt^2} + (g_0(t) + 3g_1(t)x)\frac{dx}{dt} + c_0(t) + c_1(t)x + c_2(t)x^2 + c_3(t)x^3 = 0, \tag{4.1}$$

where

$$g_1(t) = \pm\sqrt{c_3(t)}, \qquad g_0(t) = \frac{c_2(t)}{g_1(t)} - \frac{1}{2c_3(t)}\frac{dc_3}{dt}(t), \qquad c_3(t) > 0,$$

which appears in the study of interesting physical and mathematical problems [21, 22, 26, 27, 28, 34, 37].

Recently, it was found that a very general family of second-order Riccati equations admits a Lagrangian description in terms of a t-dependent non-natural regular Lagrangian

$$L(t, x, v) = \frac{1}{v + U(t, x)},$$

where $U(t, x) = a_0(t) + a_1(t)x + a_2(t)x^2$ and $a_0(t), a_1(t), a_2(t)$ are certain functions related to the t-dependent coefficients of (4.1) [27].

The Legendre transformation induced by the above Lagrangian leads to

$$p = \frac{\partial L}{\partial v} = -\frac{1}{(v + U(t, x))^2} \Longrightarrow v = \pm\frac{1}{\sqrt{-p}} - U(t, x),$$

and hence the image of the Legendre transformation is the open submanifold $\mathbb{R}\times\mathcal{O}$, where $\mathcal{O} \equiv \{(x, p) \in \mathrm{T}^*\mathbb{R} \mid p < 0\}$ (see [26] for details). If we restrict to the points (t, x, v) where $v + U(t, x) > 0$ (assuming the contrary leads to similar results), we can define in $\mathbb{R}\times\mathcal{O}$ the t-dependent Hamiltonian function

$$h(t, x, p) = vp - L(t, x, v) = p\left(\frac{1}{\sqrt{-p}} - U(t, x)\right) - \sqrt{-p} = -2\sqrt{-p} - p\,U(t, x).$$

Therefore, the Legendre transformation maps second-order Riccati equations (written as a first-order system) into the t-dependent Hamilton equations on $\mathcal{O}$ [26]:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial h}{\partial p} = \dfrac{1}{\sqrt{-p}} - a_0(t) - a_1(t)x - a_2(t)x^2, \\ \dfrac{dp}{dt} = -\dfrac{\partial h}{\partial x} = p(a_1(t) + 2a_2(t)x), \end{cases} \tag{4.2}$$

The above system is a Lie system as it describes the integral curves of the t-dependent vector field

$$X_t = X_1 - a_0(t)X_2 - a_1(t)X_3 - a_2(t)X_4,$$

where

$$X_1 = \frac{1}{\sqrt{-p}}\frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x\frac{\partial}{\partial x} - p\frac{\partial}{\partial p}, \quad X_4 = x^2\frac{\partial}{\partial x} - 2xp\frac{\partial}{\partial p},$$

along with

$$X_5 = \frac{x}{\sqrt{-p}}\frac{\partial}{\partial x} + 2\sqrt{-p}\frac{\partial}{\partial p},$$

span a five-dimensional Lie algebra of vector fields. In addition, this Lie algebra enjoys an additional property that has not been noticed so far: all their elements are Hamiltonian vector fields with respect to the Poisson bivector $\Lambda = \partial/\partial x \wedge \partial/\partial p$ on $\mathcal{O}$. Indeed, note that $X_\alpha = -\widehat{\Lambda}(dh_\alpha)$, with $\alpha = 1, \ldots, 5$ and

$$h_1(x, p) = -2\sqrt{-p}, \qquad h_2(x, p) = p, \qquad h_3(x, p) = xp, \qquad h_4(x, p) = x^2p, \\ h_5(x, p) = -2x\sqrt{-p}. \tag{4.3}$$

We can also show that second-order Kummer–Schwarz equations [10, 15], i.e. the equations

$$\frac{d^2x}{dt^2} = \frac{3}{2x}\left(\frac{dx}{dt}\right)^2 - 2c_0x^3 + 2b_1(t)x,$$

with c_0 a constant and $b_1(t)$ an arbitrary function of the time, admit similar descriptions. By using *Jacobi multipliers* [25], it can easily be derived a t-dependent non-natural Lagrangian

$$L(t,x,v) = \frac{v^2}{x^3} - 4c_0x - \frac{4b_1(t)}{x}$$

for these equations. This Lagrangian induces a Legendre transformation

$$p = \frac{2\,v}{x^3} \Longrightarrow v = \frac{p\,x^3}{2},$$

for which the induced t-dependent Hamiltonian turns out to be

$$h(t,x,p) = \frac{1}{4}p^2\,x^3 + 4c_0x + \frac{4b_1(t)}{x}.$$

Therefore, the Legendre transformation maps the Kummer–Schwarz equations (written as first-order systems) into the Hamilton equations

$$\left\{\begin{aligned} \frac{dx}{dt} &= \frac{px^3}{2}, \\ \frac{dp}{dt} &= -\frac{3p^2x^2}{4} - 4c_0 + \frac{4b_1(t)}{x^2}, \end{aligned}\right. \tag{4.4}$$

on $\mathrm{T}^*\mathbb{R}_0$, where $\mathbb{R}_0 = \mathbb{R} - \{0\}$. Once again, the above system is a Lie system as it describes the integral curves of the t-dependent vector field $X_t = X_3 + b_1(t)X_1$, where

$$X_1 = \frac{4}{x^2}\frac{\partial}{\partial p}, \quad X_2 = x\frac{\partial}{\partial x} - p\frac{\partial}{\partial p}, \quad X_3 = \frac{px^3}{2}\frac{\partial}{\partial x} - \left(\frac{3p^2x^2}{4} + 4c_0\right)\frac{\partial}{\partial p}, \tag{4.5}$$

span a three-dimensional Lie algebra V^{2KS} isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Indeed,

$$[X_1, X_3] = 2X_2, \qquad [X_1, X_2] = X_1, \qquad [X_2, X_3] = X_3.$$

Apart from providing a new approach to Kummer–Schwarz equations (see [15] for a related method), our new description possesses an additional relevant property: V^{2KS} consists of Hamiltonian vector fields with respect to the Poisson bivector $\Lambda = \partial/\partial x \wedge \partial/\partial p$ on $\mathrm{T}^*\mathbb{R}_0$. In fact, $X_\alpha = -\widehat{\Lambda}(dh_\alpha)$ with $\alpha = 1,2,3$ and

$$h_1 = \frac{4}{x}, \qquad h_2 = xp, \qquad h_3 = \frac{1}{4}p^2x^3 + 4c_0x. \tag{4.6}$$

We now focus on analysing the Hamilton equations for an n-dimensional Winternitz–Smorodinsky oscillator [57] of the form

$$\left\{\begin{aligned} \frac{dx_i}{dt} &= p_i, \\ \frac{dp_i}{dt} &= -\omega^2(t)x_i + \frac{k}{x_i^3}, \end{aligned}\right. \qquad i = 1,\ldots,n, \tag{4.7}$$

with $\omega(t)$ an arbitrary t-dependent function. These oscillators have attracted quite much attention in classical and quantum mechanics for their special properties [36, 38, 58]. In addition, observe that, when $k = 0$, Winternitz– Smorodinsky oscillators reduce to t-dependent isotropic harmonic oscillators.

System (4.7) describes the integral curves of the t-dependent vector field

$$X_t = \sum_{i=1}^{n} \left[p_i \frac{\partial}{\partial x_i} + \left(-\omega^2(t) x_i + \frac{k}{x_i^3} \right) \frac{\partial}{\partial p_i} \right]$$

on $\mathrm{T}^*\mathbb{R}_0^n$. This cotangent bundle admits a natural Poisson bivector Λ related to the restriction to $\mathrm{T}^*\mathbb{R}_0^n$ of the canonical symplectic structure on $\mathrm{T}^*\mathbb{R}^n$. If we consider the vector fields

$$\begin{gathered} X_1 = -\sum_{i=1}^{n} x_i \frac{\partial}{\partial p_i}, \qquad X_2 = \sum_{i=1}^{n} \frac{1}{2} \left(p_i \frac{\partial}{\partial p_i} - x_i \frac{\partial}{\partial x_i} \right), \\ X_3 = \sum_{i=1}^{n} \left(p_i \frac{\partial}{\partial x_i} + \frac{k}{x_i^3} \frac{\partial}{\partial p_i} \right), \end{gathered} \tag{4.8}$$

we can write $X_t = X_3 + \omega^2(t) X_1$. Additionally, since

$$[X_1, X_3] = 2X_2, \qquad [X_1, X_2] = X_1, \qquad [X_2, X_3] = X_3, \tag{4.9}$$

it follows that (4.7) is a Lie system related to a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. In addition, this Lie algebra is again made of Hamiltonian vector fields. In fact, it is easy to check that $X_\alpha = -\widehat{\Lambda}(dh_\alpha)$, with $\alpha = 1, 2, 3$ and

$$h_1 = \frac{1}{2} \sum_{i=1}^{n} x_i^2, \qquad h_2 = -\frac{1}{2} \sum_{i=1}^{n} x_i p_i, \qquad h_3 = \frac{1}{2} \sum_{i=1}^{n} \left(p_i^2 + \frac{k}{x_i^2} \right). \tag{4.10}$$

Let us now analyse a final example on a Poisson (but non-symplectic) manifold. Consider the Euler equations on the dual $\mathfrak{g}^*$ of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$, i.e.

$$\frac{d\theta}{dt} = -\mathrm{coad}_{\phi(t)} \theta, \qquad \theta \in \mathfrak{g}^*, \tag{4.11}$$

where $\phi(t)$ is a curve in $\mathfrak{g}$ and $\mathrm{coad}_{\phi(t)}\theta = -\theta \circ \mathrm{ad}_{\phi(t)} \in \mathfrak{g}^*$, which appear, for instance, in the study of geometric phases for classical systems [12, 33].

Take a basis $\{e_1, \ldots, e_r\}$ for $\mathfrak{g}$ with structure constants $c_{\alpha\beta\gamma}$, i.e. $[e_\alpha, e_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} e_\gamma$ and $\alpha, \beta = 1, \ldots, r$. It is easy to see that the vector fields $Y_\alpha(\theta) = -\mathrm{coad}_{e_\alpha}(\theta) \in \mathrm{T}_\theta \mathfrak{g}^*$, with $\alpha = 1, \ldots, r$, span a Vessiot–Guldberg Lie algebra V^E for (4.11). Indeed, they generate the Lie algebra of fundamental vector fields of the coadjoint action of a Lie group G with Lie algebra $\mathfrak{g}$ [33]. Consequently, if we write $\phi(t) = \sum_{\alpha=1}^{r} b_\alpha(t) e_\alpha$, then the Euler equations describe the integral curves of the t-dependent vector fields of the form

$$X_t^\phi = \sum_{\alpha=1}^{r} b_\alpha(t) Y_\alpha,$$

which take values in the finite-dimensional Lie algebra V^E. In other words, the Euler equations are Lie systems.

To prove that V^E consists of Hamiltonian vector fields, we need to endow $\mathfrak{g}^*$ with a Poisson structure. This can naturally be done through the so-called *Lie–Poisson bracket* on $\mathfrak{g}^*$ [47]. In fact, since $df_\theta, dg_\theta \in (\mathrm{T}_\theta \mathfrak{g}^*)^* \simeq \mathfrak{g}$ for every pair $f, g \in C^\infty(\mathfrak{g}^*)$, it makes sense to define the Lie–Poisson bracket as $\{f, g\}_{\mathfrak{g}^*}(\theta) = \langle [df_\theta, dg_\theta]_{\mathfrak{g}}, \theta \rangle$, where $\langle \cdot, \cdot \rangle$ stands for the pairing between elements of $\mathfrak{g}$ and $\mathfrak{g}^*$.

Having equipped $\mathfrak{g}^*$ with the Poisson bivector $\Lambda_{\mathfrak{g}^*}$ corresponding to the Lie–Poisson bracket, a simple calculation shows that the vector fields Y_α are Hamiltonian (with respect to $\widehat{\Lambda}_{\mathfrak{g}^*}$) with Hamiltonian functions $h_\alpha(\cdot) = -\langle e_\alpha, \cdot \rangle$.

The properties of the above relevant examples suggest us to define the following particular type of Lie systems.

Definition 4.1. We say that a system X is a *Lie–Hamilton system* if V^X is a finite-dimensional real Lie algebra of Hamiltonian vector fields with respect to a certain Poisson structure.

Note 4.2. Obsere that the above definition is equivalent to saying that X is a Lie–Hamilton system if and only if it admits a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to a certain Poisson structure.

5 Lie–Hamiltonian structures

Let us further investigate the properties of the examples provided in the previous section. Consider again the Euler equation (4.11). The Hamiltonian functions $h_\alpha(\cdot) = -\langle e_\alpha, \cdot\rangle$ of the vector fields Y_α related to Euler equations satisfy

$$\begin{aligned}\{h_\alpha, h_\beta\}_{\mathfrak{g}^*}(\theta) &= \langle [(dh_\alpha)_\theta, (dh_\beta)_\theta]_{\mathfrak{g}}, \theta\rangle = \langle [e_\alpha, e_\beta]_{\mathfrak{g}}, \theta\rangle \\ &= \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma}\langle e_\gamma, \theta\rangle = -\sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} h_\gamma(\theta).\end{aligned}$$

That is, they are a basis for a finite-dimensional real Lie algebra $(\mathfrak{W}, \{\cdot,\cdot\}_{\mathfrak{g}^*})$ of functions in $\mathfrak{g}^*$. Additionally, we can write

$$X_t^\phi = \sum_{\alpha=1}^{r} b_\alpha(t) Y_\alpha = \sum_{\alpha=1}^{r} b_\alpha(t)\widehat{\Lambda}_{\mathfrak{g}^*}(-dh_\alpha) = -\widehat{\Lambda}_{\mathfrak{g}^*}\left[d\left(\sum_{\alpha=1}^{r} b_\alpha(t) h_\alpha\right)\right].$$

In other words, the t-dependent vector field X is determined through the Poisson bivector $\Lambda_{\mathfrak{g}^*}$ and the curve $h_t = \sum_{\alpha=1}^{r} b_\alpha(t) h_\alpha$ within a finite-dimensional real Lie algebra of functions.

Likewise, the remaining examples of Section 4 enjoy a similar property. For instance, the Hamiltonian functions (4.6) and (4.10) related to the Hamilton equations (4.4) and (4.7) for second-order Kummer–Schwarz equations and Winternitz–Smorodinsky oscillators, correspondingly, satisfy the commutation relations

$$\{h_1, h_3\}_\Lambda = -2h_2, \qquad \{h_1, h_2\}_\Lambda = -h_1, \qquad \{h_2, h_3\}_\Lambda = -h_3, \tag{5.1}$$

where $\{\cdot,\cdot\}_\Lambda$ stands for the Poisson structure associated to the Poisson bivector Λ of each example. The t-dependent vector fields governing the dynamics of systems (4.4) and (4.7) can therefore be written in the form $X_t = -\widehat{\Lambda}\circ d(h_3 + d(t)h_1)$, where $d(t), h_1, h_2, h_3$ are the corresponding functions for each problem, e.g. $d(t) = \omega^2(t)$ and h_1, h_2, h_3 given by (4.10) for the system (4.7). This leads us to define the new following notions.

Definition 5.1. A *Lie–Hamiltonian structure* is a triple (N, Λ, h), where (N, Λ) stands for a Poisson manifold and h represents a t-parametrised family of functions $h_t : N \to \mathbb{R}$ such that $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\Lambda)$ is a finite-dimensional real Lie algebra.

Definition 5.2. A t-dependent vector field X is said to admit, or to possess, a Lie–Hamiltonian structure (N, Λ, h) if $X_t = -\widehat{\Lambda}\circ dh_t$ for all $t \in \mathbb{R}$.

Proposition 5.3. *If a system X admits a Lie–Hamiltonian structure, then X is a Lie–Hamilton system.*

Proof. Let (N, Λ, h) be a Lie–Hamiltonian structure for X. In consequence, $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$ is a finite-dimensional Lie algebra. Moreover, $\{X_t\}_{t\in\mathbb{R}} \subset \widehat{\Lambda}\circ d[\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})]$, and as $\widehat{\Lambda}\circ d$ is a Lie algebra morphism, it follows that $V = \widehat{\Lambda}\circ d[\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})]$ is a finite-dimensional Lie algebra of Hamiltonian vector fields containing $\{X_t\}_{t\in\mathbb{R}}$. Therefore, $V^X \subset V$ and X is a Lie–Hamilton system. □

Observe that every Lie–Hamiltonian structure (N, Λ, h) induces a unique Lie–Hamilton system $X_t = -\widehat{\Lambda} \circ dh_t$ admitting it as a Lie–Hamiltonian structure. This is interesting as Lie–Hamiltonian structures appear in the physics literature and they therefore allow us to determine Lie–Hamilton systems of interest [6, 7, 8, 9]. For instance, consider a Lie algebra morphism $D : (\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}) \to (C^\infty(T^*M), \{\cdot,\cdot\})$, where $\mathfrak{g}$ is a finite-dimensional real Lie algebra and $\{\cdot,\cdot\}$ is the canonical Poisson structure on T^*M defined by its natural symplectic structure. This is the case when we have a strongly Hamiltonian action of G on T^*M (see [44]), i.e. a comomentum map which is additionally a Lie algebra homomorphism. When choosing a basis $e_1, \ldots, e_r$ for $\mathfrak{g}$, we can define a t-dependent Hamiltonian of the form

$$h_t = \sum_{\alpha=1}^{r} b_\alpha(t) D(e_\alpha). \tag{5.2}$$

As the Lie algebra $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\})$ is included in the finite-dimensional real Lie algebra $D(\mathfrak{g})$, then $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$ is finite-dimensional and for every curve $h_t \subset \mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$ the triple (N, Λ, h) is a Lie–Hamiltonian structure. Hamiltonians of the form (5.2) appear in the physics literature [4, 6, 7, 8, 9, 42, 43] and, in view of Proposition 5.3, they give rise to new Lie–Hamilton systems that can be studied through our techniques.

Let us now analyse the relations between a system V^X and the Lie algebra $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\Lambda)$ for a system X admitting a Lie–Hamiltonian structure (N, Λ, h).

Lemma 5.4. *Given a system X on N possessing an Lie–Hamiltonian structure (N, Λ, h), we have that*

$$0 \hookrightarrow \mathrm{Cas}(N, \Lambda) \cap \mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}) \hookrightarrow \mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}) \xrightarrow{\mathcal{J}_\Lambda} V^X \to 0, \tag{5.3}$$

where $\mathcal{J}_\Lambda : f \in \mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}) \mapsto \widehat{\Lambda} \circ df \in V^X$, is an exact sequence of Lie algebras.

Proof. Consider the exact sequence of (generally) infinite-dimensiaonl real Lie algebras

$$0 \hookrightarrow \mathrm{Cas}(N, \Lambda) \hookrightarrow C^\infty(N) \xrightarrow{\widehat{\Lambda}\circ d} \mathrm{Ham}(N, \Lambda) \to 0.$$

Since $X_t = -\widehat{\Lambda} \circ dh_t$, we see that $V^X = \mathrm{Lie}(\widehat{\Lambda} \circ d(\{h_t\}_{t\in\mathbb{R}}))$. Using that $\widehat{\Lambda} \circ d$ is a Lie algebra morphism, we have $V^X = \widehat{\Lambda} \circ d[\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})] = \mathcal{J}_\Lambda(\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}))$. Additionally, as $\mathcal{J}_\Lambda$ is the restriction to $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$ of $\widehat{\Lambda} \circ d$, we obtain that its kernel consists of Casimir functions belonging to $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$, i.e. $\ker \mathcal{J}_\Lambda = \mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}) \cap \mathrm{Cas}(N, \Lambda)$. The exactness of sequence (5.3) easily follows from these results. □

The above proposition entails that every system X that possesses a Lie–Hamiltonian structure (N, Λ, h) is such that $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$ is a Lie algebra extension of V^X by $\mathrm{Cas}(N, \Lambda) \cap \mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$, i.e. the sequence of Lie algebras (5.3) is exact. Note that if X is a Lie system, all the Lie algebras appearing in such a sequence are finite-dimensional. For instance, the first-order system (4.2) associated to second-order Riccati equations admits a Lie–Hamiltonian structure

$$\left(\mathcal{O}, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial p}, h_1 - a_0(t) h_2 - a_1(t) h_3 - a_2(t) h_4\right),$$

where $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$, for generic functions $a_0(t), a_1(t), a_2(t)$, is a six-dimensional Lie algebra of functions $\mathfrak{W} \simeq V^X \oplus \mathbb{R}$.

It is worth noting that every t-dependent vector field that admits a Lie–Hamiltonian structure necessarily possesses many other Lie–Hamiltonian structures. For instance, if system X admits (N, Λ, h), then it also admits a Lie–Hamiltonian structure (N, Λ, h'), with $h' : (t, x) \in \mathbb{R} \times N \mapsto h(t, x) + f_{\mathcal{C}}(x) \in \mathbb{R}$, where $f_{\mathcal{C}}$ is any Casimir function with respect to Λ. Indeed, it is easy to see that if $h_1, \ldots, h_r$ is a basis for $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$, then $h_1, \ldots, h_r, f_{\mathcal{C}}$ span $\mathrm{Lie}(\{h'_t\}_{t\in\mathbb{R}})$, which also becomes a finite-dimensional real Lie algebra. As shown later, this has relevant implications for the linearisation of Lie–Hamilton systems.

We have already proved that every system X admitting a Lie–Hamiltonian structure must possess several ones. Nevertheless, we have not yet studied the conditions ensuring that a Lie–Hamilton system X possesses a Lie–Hamiltonian structure. Let us answer this question.

Proposition 5.5. *Every Lie–Hamilton system admits a Lie–Hamiltonian structure.*

Proof. Assume X to be a Lie–Hamilton system on a manifold N with respect to a Poisson bivector Λ. Since $V^X \subset \mathrm{Ham}(N,\Lambda)$ is finite-dimensional, there exists a finite-dimensional linear space $\mathfrak{W}_0 \subset C^\infty(N)$ isomorphic to V^X and such that $\widehat{\Lambda}\circ d(\mathfrak{W}_0) = V^X$. Consequently, there exists a curve h_t in $\mathfrak{W}_0$ such that $X_t = -\widehat{\Lambda}\circ d(h_t)$. To ensure that h_t gives rise to a Lie–Hamiltonian structure, we need to demonstrate that $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\Lambda)$ is finite-dimensional. This will be done by constructing a finite-dimensional Lie algebra of functions containing the curve h_t.

Define the linear isomorphism $T : X_f \in V^X \mapsto -f \in \mathfrak{W}_0 \subset C^\infty(N)$ associating each vector field in V^X with minus its unique Hamiltonian function within $\mathfrak{W}_0$. This can be done by choosing a representative for each element of a basis of V^X and extending the map by linearity.

Note that this mapping needs not be a Lie algebra morphism and hence $\mathrm{Im}\, T = \mathfrak{W}_0$ does not need to be a Lie algebra. Indeed, we can define a bilinear map $\Upsilon : V^X \times V^X \to C^\infty(N)$ of the form

$$\Upsilon(X_f, X_g) = \{f, g\}_\Lambda - T[X_f, X_g], \tag{5.4}$$

measuring the obstruction for T to be a Lie algebra morphism, i.e. Υ is identically null if and only if T is a Lie algebra morphism. In fact, if $\mathfrak{W}_0$ were a Lie algebra, then $\{f, g\}_\Lambda$ would be the only element of $\mathfrak{W}_0$ with Hamiltonian vector field $-[X_f, X_g]$, i.e. $T[X_f, X_g]$, and Υ would be a zero function.

Note that $\Upsilon(X_f, X_g)$ is the difference between two functions, namely $\{f, g\}_\Lambda$ and $T[X_f, X_g]$, sharing the same Hamiltonian vector field. Consequently, $\mathrm{Im}\,\Upsilon \subset \mathrm{Cas}(N,\Lambda)$ and it can be injected into a finite-dimensional Lie algebra of Casimir functions of the form

$$\mathfrak{W}_\mathcal{C} \equiv \langle \Upsilon(X_i, X_j)\rangle, \qquad i, j = 1, \ldots, r,$$

where $X_1, \ldots, X_r$ is a basis for V^X. From here, it follows that

$$\{\mathfrak{W}_\mathcal{C}, \mathfrak{W}_\mathcal{C}\}_\Lambda = 0, \quad \{\mathfrak{W}_\mathcal{C}, \mathfrak{W}_0\}_\Lambda = 0, \quad \{\mathfrak{W}_0, \mathfrak{W}_0\}_\Lambda \subset \mathfrak{W}_\mathcal{C} + \mathfrak{W}_0.$$

Hence, $\mathfrak{W} \equiv \mathfrak{W}_0 + \mathfrak{W}_\mathcal{C}$ is a finite-dimensional Lie algebra of functions containing the curve h_t. From here, it readily follows that X admits a Lie–Hamiltonian structure $(N, \Lambda, -TX_t)$. □

Since every Lie–Hamilton system possesses a Lie–Hamiltonian structure and every Lie–Hamiltonian structure determine a Lie–Hamilton systems, we obtain the following theorem.

Theorem 5.6. *A system X admits a Lie–Hamiltonian structure if and only if it is a Lie–Hamilton system.*

6 On general properties of Lie–Hamilton systems

We now turn to describing the analogue for Lie–Hamilton systems of the basic properties of general Lie systems. Additionally, we show how the Poisson structures associated to Lie–Hamilton systems allow us to investigate their t-independent constants of motion, Lie symmetries, superposition rules and linearisation properties.

Recall that, as for every Lie system, the general solution $x(t)$ of a Lie–Hamilton system X on N can be brought into the form $x(t) = \varphi(g(t), x_0)$, where $x_0 \in N$, the map $\varphi : G \times N \to N$ is the action of a connected Lie group G whose space of fundamental vector fields is $V^X \simeq T_eG$, and $g(t)$ is the solution of a Lie system of the form (3.1). In addition, for a Lie–Hamilton system, the infinitesimal action associated to φ, let us say $\rho^X : \mathfrak{g} \to \Gamma(\tau_N)$, takes also values in a certain space $\mathrm{Ham}(N,\Lambda)$. In other words, φ is a *Hamiltonian* Lie group action. Furthermore, the mappings $\varphi_g : x \in N \mapsto \varphi(g, x) \in N$, with $g \in G$, are *Poisson maps*, i.e.

$$\varphi_{g*}\Lambda = \Lambda.$$

The above Lie group action plays another relevant rôle. It is known that if G is connected, every curve $\bar{g}(t)$ in G induces a t-dependent change of variables mapping a Lie system X taking values in a Lie algebra V^X into another Lie system $\bar{X}$, with general solution $\bar{x}(t) = \varphi(\bar{g}(t), x(t))$, taking values in the same Lie algebra V^X [14, 18, 20]. In the particular case of X being a Lie–Hamilton system, the vector fields $\{\bar{X}_t\}_{t\in\mathbb{R}}$ are also Hamiltonian and $\bar{X}$ is again a Lie–Hamilton system.

Using again that $x(t) = \varphi(g(t), x_0)$, we see that the solutions of a Lie system X are contained in the orbits of φ. Indeed, it is easy to see that the vector fields $\{X_t\}_{t\in\mathbb{R}}$ are tangent such orbits. Therefore, the integration of a Lie system X reduces to integrating its restrictions to each orbit of φ, which are Lie systems also.

Meanwhile, for Lie–Hamilton systems, we have another related method of reduction. Note that given a Lie–Hamilton system X admitting a Lie–Hamilton structure (N, Λ, h), we have that $\mathcal{D}^X \subset \mathcal{F}^\Lambda$, where we recall that $\mathcal{F}^\Lambda$ is the characteristic distribution related to Λ and $\mathcal{D}^X$ is spanned by Hamiltonian vector fields within V^X. Hence, the vector fields $\{X_t\}_{t\in\mathbb{R}}$ are tangent to the symplectic leaves of the Poisson manifold (N, Λ). From here, it immediately follows the theorem below.

Theorem 6.1. *The integration of a Lie–Hamilton system X possessing a Lie–Hamiltonian structure (N, Λ, h) reduces to integrating the restrictions of $X|_{\mathfrak{F}^\Lambda}$ to every symplectic leaf $\mathfrak{F}^\Lambda$ associated to (N, Λ). Every such a system is a Lie–Hamilton system with respect to the symplectic structure $(\mathfrak{F}^\Lambda, \omega_{\mathfrak{F}^\Lambda})$ induced by the restriction of Λ to $\mathfrak{F}^\Lambda$.*

Let us now turn to describing several properties of constants of motion for Lie systems.

Proposition 6.2. *Given a system X with a Lie–Hamiltonian structure (N, Λ, h), then $\mathcal{C}^\Lambda \subset \mathcal{V}^X$, where we recall that $\mathcal{C}^\Lambda$ is the Casimir distribution relative to Λ.*

Proof. Consider a $\theta_x \in \mathcal{C}^\Lambda_x$, with $x \in N$. As X is a Lie–Hamilton system, for every $Y \in V^X$ there exists a function $f \in C^\infty(N)$ such that $Y = -\widehat{\Lambda}(df)$. Then,

$$\theta_x(Y_x) = -\theta_x(\widehat{\Lambda}_x(df_x)) = -\Lambda_x(df_x, \theta_x) = 0,$$

where $\widehat{\Lambda}_x$ is the restriction of $\widehat{\Lambda}$ to T^*_xN. As the vectors Y_x, with $Y \in V^X$, span $\mathcal{D}^X_x$, then $\theta_x \in \mathcal{V}^X_x$ and $\mathcal{C}^\Lambda \subset \mathcal{V}^X$. □

Observe that different Lie–Hamiltonian structures for a Lie–Hamilton system X may lead to different families of Casimir functions, which may determine different constants of motion for X.

Theorem 6.3. *Let X be a system admitting a Lie–Hamiltonian structure (N,Λ, h), the space $\mathcal{I}^X|_U$ of t-independent constants of motion of X on an open $U \subset U^X$ is a Poisson algebra. Additionally, the codistribution $\mathcal{V}^X|_{U_X}$ is involutive with respect to the Lie bracket $[\cdot, \cdot]_\Lambda$ induced by Λ on $\Gamma(\pi_N)$.*

Proof. Let $f_1, f_2 : U \to \mathbb{R}$ be two t-independent functions constants of motion for X, i.e. $X_t f_i = 0$, for $i = 1, 2$ and $t \in \mathbb{R}$. As X is a Lie–Hamilton system, all the elements of V^X are Hamiltonian vector fields and we can write $Y\{f, g\}_\Lambda = \{Y f, g\}_\Lambda + \{f, Y g\}_\Lambda$ for every $f, g \in C^\infty(N)$. In particular, $X_t(\{f_1, f_2\}_\Lambda) = \{X_t f_1, f_2\}_\Lambda + \{f_1, X_t f_2\}_\Lambda = 0$, i.e. the Poisson bracket of t-independent constants of motion is a new one. As $\lambda f_1 + \mu f_2$ and $f_1 \cdot f_2$ are also t-independent constants of motion for every $\lambda, \mu \in \mathbb{R}$, it easily follows that $\mathcal{I}^X|_U$ is a Poisson algebra.

In view of Lemma 3.4, the co-distribution $\mathcal{V}^X$ admits a local basis of exact forms $df_1, \ldots, df_{p(x)}$ for every point $x \in U^X$, where $\mathcal{V}^X$ has local constant rank $p(x) = \dim N - \dim \mathcal{D}^X_x$. Now, $[df_i, df_j]_\Lambda = d(\{f_i, f_j\}_\Lambda)$ for $i, j = 1, \ldots, p(x)$. We already proved that the function $\{f_i, f_j\}_\Lambda$ is another first-integral. Therefore, in view of Lemma 3.4, it easily follows that $\{f_i, f_j\}_\Lambda = G(f_1, \ldots, f_{p(x)})$. Thus, $[df_i, df_j]_\Lambda \in \mathcal{V}^X|_{U_X}$. From here and using the properties of the Lie bracket $[\cdot, \cdot]_\Lambda$, it directly turns out that the Lie bracket of two one-forms taking values in $\mathcal{V}^X|_{U_X}$ belongs to $\mathcal{V}^X|_{U_X}$. Hence, $\mathcal{V}^X|_{U_X}$ is involutive. □

Corollary 6.4. *Given a Lie–Hamilton system X, the space $\mathcal{I}^X|_U$, where $U \subset U^X$ is such that $\mathcal{V}^X$ admits a local basis of exact forms, is a function group, that is:*

1. *The space $\mathcal{I}^X|_U$ is a Poisson algebra.*

2. *There exists a family of functions $f_1, \dots, f_s \in \mathcal{I}^X|_U$ such that every element f of $\mathcal{I}^X|_U$ can be put in the form $f = F(f_1, \dots, f_s)$ for a certain function $F : \mathbb{R}^s \to \mathbb{R}$.*

Proof. In view of the previous theorem, $\mathcal{I}^X|_U$ is a Poisson algebra with respect to a certain Poisson bracket. Taking into account Proposition 3.3 and the form of $\mathcal{I}^X|_U$ given by Lemma 3.4, we obtain that this space becomes a function group. □

The above properties do not necessarily hold for systems other than Lie–Hamilton systems, as they do not need to admit any *a priori* relation among a Poisson bracket of functions and the t–dependent vector field describing the system. Let us exemplify this. Consider the Poisson manifold $(\mathbb{R}^3, \Lambda_{GM})$, where

$$\Lambda_{GM} = \sigma_3 \frac{\partial}{\partial \sigma_2} \wedge \frac{\partial}{\partial \sigma_1} - \sigma_1 \frac{\partial}{\partial \sigma_2} \wedge \frac{\partial}{\partial \sigma_3} + \sigma_2 \frac{\partial}{\partial \sigma_3} \wedge \frac{\partial}{\partial \sigma_1}$$

and $(\sigma_1, \sigma_2, \sigma_3)$ is a coordinate basis for $\mathbb{R}^3$, appearing in the study of Classical XYZ Gaudin Magnets [9]. The system $X = \partial/\partial\sigma_3$ is not a Lie–Hamilton system with respect to this Poisson structure as X is not Hamiltonian, namely $\mathcal{L}_X \Lambda_{GM} \neq 0$. In addition, this system admits two first-integrals σ_1 and σ_2. Nevertheless, their Lie bracket reads $\{\sigma_1, \sigma_2\} = -\sigma_3$, which is not a first-integral for X. On the other hand, consider the system

$$Y = \sigma_3 \frac{\partial}{\partial \sigma_2} + \sigma_2 \frac{\partial}{\partial \sigma_3}.$$

This system is a Lie–Hamilton system, as it can be written in the form $Y = -\widehat{\Lambda}_{GM}(d\sigma_1)$, and it possesses two first-integrals given by σ_1 and $\sigma_2^2 - \sigma_3^2$. Unsurprisingly, $Y\{\sigma_1, \sigma_2^2 - \sigma_3^2\} = 0$, i.e. the Lie bracket of two t-independent constants of motion is also a constant of motion.

Let us prove some final interesting results about the t-independent constants of motion for Lie–Hamilton systems.

Proposition 6.5. *Let X be a Lie–Hamilton system that admits a Lie–Hamiltonian structure (N, Λ, h). The function $f : N \to \mathbb{R}$ is a constant of motion for X if and only if f Poisson commutes with all elements of* $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\Lambda)$.

Proof. The function f is a t-independent constant of motion for X if and only if

$$0 = X_t f = \{f, h_t\}_\Lambda, \qquad \forall t \in \mathbb{R}. \tag{6.1}$$

From here,

$$\{f, \{h_t, h_{t'}\}_\Lambda\}_\Lambda = \{\{f, h_t\}_\Lambda, h_{t'}\}_\Lambda + \{h_t, \{f, h_{t'}\}_\Lambda\}_\Lambda = 0, \qquad \forall t, t' \in \mathbb{R},$$

and inductively follows that f Poisson commutes with all successive Poisson brackets of elements of $\{h_t\}_{t\in\mathbb{R}}$ and their linear combinations. As these elements span $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$, we get that f Poisson commutes with $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$.

Conversely, if f Poisson commutes with $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$, it Poisson commutes with the elements $\{h_t\}_{t\in\mathbb{R}}$, and, in view of (6.1), it becomes a constant of motion for X. □

In order to illustrate the above proposition, let us consider a Winternitz–Smorodinsky system (4.7) with $n = 2$. Recall that this system admits a Lie–Hamiltonian structure $(\mathrm{T}^*\mathbb{R}^2_0, \Lambda, h = h_3 + \omega^2(t)h_1)$, where $\Lambda = \sum_{i=1}^2 \partial/\partial x_i \wedge \partial/\partial p_i$ is a Poisson bivector on $\mathrm{T}^*\mathbb{R}^2_0$ and the functions h_1, h_3 are given within (4.10). For non-constant $\omega(t)$, it is easy to prove that $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\Lambda)$ is a real Lie algebra of functions isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ generated by the functions h_1, h_2 and h_3 detailed in (4.10). When $\omega(t) = \omega_0 \in \mathbb{R}$, the Lie algebra $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\Lambda)$ becomes a one-dimensional Lie subalgebra of the previous one. In any case, it is known that

$$I = (x_1 p_2 - p_1 x_2)^2 + k \left[\left(\frac{x_1}{x_2} \right)^2 + \left(\frac{x_2}{x_1} \right)^2 \right] \tag{6.2}$$

is a t-independent constant of motion (cf. [24]). A simple calculation shows that

$$\{I, h_\alpha\}_\Lambda = 0, \qquad \alpha = 1, 2, 3.$$

Then, the function I always Poisson commutes with the whole Lie algebra $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\Lambda)$, as expected.

Obviously, every autonomous Hamiltonian system is a Lie–Hamilton system possessing a Lie–Hamiltonian structure (N, Λ, h) with h being a time-independent Hamiltonian. Consequently, above proposition shows that the time-independent first-integrals for a Hamiltonian system are those functions that Poisson commute with its Hamiltonian, recovering as a particular case this wide-known result.

Moreover, above proposition suggests us that the rôle played by autonomous Hamiltonians for Hamiltonian systems is performed by the finite-dimensional Lie algebras of functions associated with Lie–Hamiltonian structures in the case of Lie–Hamilton systems. This can be employed, for instance, to study time-independent first-integrals of Lie–Hamilton systems or, more specifically, the maximal number of such first-integrals in involution, which would lead to the interesting analysis of integrability/superintegrability of Lie–Hamilton systems.

Definition 6.6. We say that a Lie system X admitting a Lie–Hamilton structure (N, Λ, h) possesses a compatible *strong comomentum map* with respect to this Lie–Hamilton structure if there exists a Lie algebra morphism $\lambda : V^X \to \mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\Lambda)$ such that the following diagram:

$$\begin{array}{ccc} & & V^X \\ & \overset{\lambda}{\swarrow} & \big\downarrow \iota \\ \mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\Lambda) & \xrightarrow{\widehat{\Lambda}\circ d} & \mathrm{Ham}(N, \Lambda) \end{array}$$

where $\iota : V^X \hookrightarrow \mathrm{Ham}(N, \Lambda)$ is the natural injection of V^X into $\mathrm{Ham}(N, \Lambda)$, is commutative.

Observe that, in this case, X induces a Hamiltonian Lie group action $\varphi : G \times N \to N$ whose set of fundamental vector fields lies in V^X and admits a comomentum map $\lambda : V^X \to \mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\Lambda) \subset C^\infty(N)$ that is a Lie algebra morphism, i.e. we say that φ is a *strongly Hamiltonian action*. Note additionally that given $Y \in V^X$, we have $\widehat{\Lambda} \circ d \circ \lambda(Y) = Y$, i.e. $-\lambda(Y)$ is a Hamiltonian function for Y.

Conversely, it can easily be proved that if X is a Lie system inducing such a strongly Hamiltonian Lie group action, then X is a Lie–Hamilton system admitting a Lie–Hamiltonian structure $(N, \Lambda, -\lambda(X_t))$ that is compatible with a strong comomentum map λ.

It is important to note that if X possesses a Lie–Hamiltonian structure (N, Λ, h) compatible with a strong comomentum map λ, then the Lie algebra $\lambda(V^X)$ is isomorphic to V^X and it can readily be proved that X admits an additional Lie–Hamiltonian structure $(N, \Lambda, \bar{h}_t \equiv -\lambda(X_t))$ satisfying that $V^X \simeq \mathrm{Lie}(\{\bar{h}_t\}_{t\in\mathbb{R}})$ and admitting λ as a compatible strong comomentum map.

Let us provide a particular example of a strong comomentum map for a second-order Kummer–Schwarz equation in Hamiltonian form (4.4) with a non-constant $\omega(t)$. Since the corresponding t-dependent vector field X satisfies that $X_t = X_3 + \omega^2(t)X_1$, where X_1, X_3 are given by (4.5), and in view of the relations (4.9), the Lie algebra V^X is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Recall that this system admits a Lie–Hamiltonian structure $(\mathrm{T}^*\mathbb{R}_0, \Lambda, h_t = h_3 + \omega^2(t)h_1)$, with h_1 and h_3 given in (4.5). It is easy to see that X admits a strong comomentum map, relative to the previous Lie–Hamiltonian structure, $\lambda : V^X \to C^\infty(\mathrm{T}^*\mathbb{R}_0)$ such that $\lambda(X_\alpha) = -h_\alpha$, where X_1, X_2, X_3 and h_1, h_2, h_3 are given by (4.5) and (4.6), correspondingly. As expected, V^X and $\lambda(V^X) = \mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\Lambda)$ are isomorphic Lie algebras. A similar result can readily be obtained for $\omega(t) = \omega_0$, with $\omega_0 \in \mathbb{R}$. In this case, we now have that V^X and the related $\mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\Lambda)$ become one-dimensional Lie algebras.

Proposition 6.7. *Let X be a Lie system possessing a Lie–Hamiltonian structure (N, Λ, h) compatible with a strong comomentum map λ such that $\dim \mathcal{D}^X_x = \dim N = \dim V^X$ at a generic $x \in N$. Then, there exists a local coordinate system defined on a neighbourhood of each x such that X and Λ are simultaneously linearisable and where X possesses a linear superposition rule.*

Proof. As it is assumed that $n \equiv \dim N = \dim V^X = \dim \mathcal{D}_x^X$ at a generic x, every basis $X_1, \ldots, X_n$ of V^X gives rise to a basis for the tangent bundle TN on a neighbourhood of x. Since X admits a strong comomentum map compatible with (N, Λ, h), we have $(V^X, [\cdot,\cdot]) \simeq (\lambda(V^X), \{\cdot,\cdot\}_\Lambda)$ and the family of functions, $h_\alpha = \lambda(X_\alpha)$, with $\alpha = 1, \ldots, n$, form a basis for the Lie subalgebra $\lambda(V^X)$. Moreover, since $\widehat{\Lambda} \circ d \circ \lambda(V^X) = V^X$ and dim $V^X = \dim \mathcal{D}_{x'}^X$ for x' in a neighbourhood of x, then $\widehat{\Lambda}_{x'} \circ d(\lambda(V^X)) \simeq T_{x'}N$ and $dh_1 \wedge \ldots \wedge dh_n \neq 0$ at a generic point. Hence, the set $(h_1, \ldots, h_n)$ is a coordinate system on an open dense subset of N. Now, using again that $(\lambda(V^X), \{\cdot,\cdot\}_\Lambda)$ is a real Lie algebra, the Poisson bivector Λ can be put in the form

$$\Lambda = \frac{1}{2}\sum_{i,j=1}^{n} \{h_i, h_j\}_\Lambda \frac{\partial}{\partial h_i} \wedge \frac{\partial}{\partial h_j} = \frac{1}{2}\sum_{i,j,k=1}^{n} c_{ijk} h_k \frac{\partial}{\partial h_i} \wedge \frac{\partial}{\partial h_j}, \tag{6.3}$$

for certain real n^3 constants c_{ijk}. In other words, the Poisson bivector Λ becomes linear in the chosen coordinate system.

Since we can write $X_t = -\widehat{\Lambda}(d\bar{h}_t)$, with $\bar{h}_t = -\lambda(X_t)$ being a curve in the Lie algebra $\lambda(V^X) \subset \mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}})$, expression (6.3) yields

$$\begin{aligned} X_t = -\widehat{\Lambda}(d\bar{h}_t) &= -\widehat{\Lambda} \circ d\left(\sum_{l=1}^{n} b_l(t) h_l\right) \\ &= -\sum_{l=1}^{n} b_l(t)(\widehat{\Lambda} \circ dh_l) = -\sum_{l,j,k=1}^{n} b_l(t) c_{ljk} h_k \frac{\partial}{\partial h_j}, \end{aligned}$$

and X_t is linear in this coordinate system. Consequently, as every linear system, X admits a linear superposition rule in the coordinate system $(h_1, \ldots, h_n)$. □

Let us turn to describing some features of t-independent Lie symmetries for Lie–Hamilton systems. Our exposition will be based upon the properties of the hereafter called *symmetry distribution*.

Definition 6.8. Given a Lie–Hamilton system X that possesses a Lie–Hamiltonian structure (N, Λ, h), we define its *symmetry distribution*, $\mathcal{S}_\Lambda^X$, by

$$(\mathcal{S}_\Lambda^X)_x = \widehat{\Lambda}_x(\mathcal{V}_x^X) \in T_xN, \qquad x \in N.$$

As its name indicates, the symmetry distribution can be employed to investigate the t-independent Lie symmetries of a Lie–Hamilton system. Let us give some basic examples of how this can be done.

Proposition 6.9. *Given a Lie–Hamilton system X with a Lie–Hamiltonian structure (N, Λ, h), then:*

1. *The symmetry distribution $\mathcal{S}_\Lambda^X$ associated with X and Λ is involutive on U^X, i.e. the open dense subset of N where $\mathcal{V}^X$ is differentiable.*

2. *If f is a t-independent constant of motion for X, then $\widehat{\Lambda}(df)$ is a t-independent Lie symmetry of X.*

3. *The distribution $\mathcal{S}_\Lambda^X$ admits a local basis of t-independent Lie symmetries of X defined around a generic point of N. The elements of such a basis are Hamiltonian vector fields of t-independent constants of motion of X.*

Proof. By definition of $\mathcal{S}_\Lambda^X$ and using that $\mathcal{V}^X$ has constant rank on the connected components of U_X, we can ensure that given two vector fields in $Y_1, Y_2 \in \mathcal{S}_\Lambda^X|_{U_X}$, there exist two forms $\omega, \omega' \in \mathcal{V}^X|_{U_X}$ such that $Y_1 = \widehat{\Lambda}(\omega)$, $Y_2 = \widehat{\Lambda}(\omega')$. Since X is a Lie–Hamilton system, $\mathcal{V}^X|_{U_X}$ is involutive and $\widehat{\Lambda}$ is an anchor, i.e. a Lie algebra morphism from $(\Gamma(\pi_N), [\cdot,\cdot]_\Lambda)$ to $(\Gamma(\tau_N), [\cdot,\cdot])$, then

$$[Y_1, Y_2] = [\widehat{\Lambda}(w), \widehat{\Lambda}(w')] = \widehat{\Lambda}([w, w']_\Lambda) \in \mathcal{S}_\Lambda^X.$$

In other words, since $\mathcal{V}^X$ is involutive on U_X, then $\mathcal{S}^X_\Lambda$ is so, which proves (1).

To prove (2), note that

$$[X_t, \widehat{\Lambda}(df)] = -[\widehat{\Lambda}(dh_t), \widehat{\Lambda}(df)] = -\widehat{\Lambda}(d\{h_t, f\}_\Lambda) = \widehat{\Lambda}[d(X_t f)] = 0.$$

Finally, the proof of (3) is based upon the fact that $\mathcal{V}^X$ admits, around a point $x \in U^X \subset N$, a local basis of one-forms $df_1, \ldots, df_{p(x)}$, with $f_1, \ldots, f_{p(x)}$ being a family of t-independent constants of motion for X and $p(x) = \dim N - \dim \mathcal{D}^X_x$. From (2), the vector fields $X_{f_1}, \ldots, X_{f_{p(x)}}$ form a family of Lie symmetries of X locally spanning $\mathcal{S}^X_\Lambda$. Hence, we can easily choose among them a local basis for $\mathcal{S}^X_\Lambda$. □

As a particular example of the usefulness of the above result, let us turn to a two-dimensional Winternitz–Smorodinsky oscillator X given by (4.7) and its known constant of motion (6.2). In view of the previous proposition, $Y = \widehat{\Lambda}(dI)$ must be a Lie symmetry for these systems. A little calculation leads to

$$Y = 2(x_1p_2 - p_1x_2)\left(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}\right) + \\ 2\left[(x_1p_2 - p_1x_2)p_2 + k\frac{x_1^4 - x_2^4}{x_1^3x_2^2}\right]\frac{\partial}{\partial p_1} - 2\left[(x_1p_2 - p_1x_2)p_1 + k\frac{x_1^4 - x_2^4}{x_2^3x_1^2}\right]\frac{\partial}{\partial p_2},$$

and it is straightforward to verify that Y commutes with X_1, X_2, X_3, given by (4.8), and therefore with every X_t, with $t \in \mathbb{R}$, i.e. Y is a Lie symmetry for X.

Proposition 6.10. *Let X be a Lie–Hamilton system possessing a Lie–Hamiltonian structure (N, Λ, h). If $[V^X, V^X] = V^X$ and $Y \in \mathrm{Ham}(N, \Lambda)$ is a Lie symmetry of X, then $Y \in \mathcal{S}^X_\Lambda$.*

Proof. As Y is a t-independent Lie symmetry, then $[Y, X_t] = 0$ for every $t \in \mathbb{R}$. Since Y is a Hamiltonian vector field, then $Y = -\widehat{\Lambda} \circ df$ for a certain $f \in C^\infty(N)$. Using that $X_t = -\widehat{\Lambda}(dh_t)$, we obtain

$$0 = [Y, X_t] = [\widehat{\Lambda}(df), \widehat{\Lambda}(dh_t)] = \widehat{\Lambda}(d\{f, h_t\}_\Lambda) = \widehat{\Lambda}[d(X_t f)].$$

Hence, $X_t f$ is a Casimir function. Therefore, as every $X_{t'}$ is a Hamiltonian vector field for all $t' \in \mathbb{R}$, it turns out that $X_{t'}X_t f = 0$ for every $t, t' \in \mathbb{R}$ and, in consequence, $Z_1 f$ is a Casimir function for every $Z_1 \in V^X$. Moreover, as every $Z_2 \in V^X$ is Hamiltonian, we have

$$Z_2Z_1f = Z_1Z_2f = 0 \Longrightarrow (Z_2Z_1 - Z_1Z_2)f = [Z_2, Z_1]f = 0.$$

As $[V^X, V^X] = V^X$, every element Z of V^X can be written as the commutator of two elements of V^X and, in view of the above expression, $Zf = 0$ which shows that f is a t-independent constant of motion for X. Finally, as $Y = -\widehat{\Lambda}(df)$, then $Y \in \mathcal{S}^X_\Lambda$. □

Note that, roughly speaking, the above proposition ensures that, when V^X is *perfect*, i.e. $[V^X, V^X] = V^X$ (see [13]), then $\mathcal{S}^X_\Lambda$ contains all Hamiltonian Lie symmetries of X. This is the case for Winternitz–Smorodinsky systems (4.7) with a non-constant $\omega(t)$, whose V^X was already shown to be isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

7 Conclusions and Outlook

We have laid down the background for the analysis of a class of systems of first-order ordinary differential equations whose dynamic is determined by a curve in a Lie algebra of Hamiltonian vector fields, i.e. the Lie–Hamilton systems. We proved that these systems can be described through curves in finite-dimensional real Lie algebras of functions on a Poisson manifold, the Lie–Hamiltonian structures. Such structures have been employed to study features of Lie–Hamilton systems, e.g. linearisability conditions, constants of motion, Lie symmetries, superposition rules, etc. All our methods and results have been illustrated by examples of mathematical and physical interest.

Apart from the results derived within this work, there remains a big deal of further properties to be analysed: the existence of several Lie–Hamiltonian structures for a system, the study of conditions for the existence of Lie–Hamilton systems, methods to derive superposition rules, the analysis of integrable and superintegrable Lie–Hamilton systems, etc. We plan to investigate all these topics in the future.

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