Int J Theor Phys (2011) 50:2046–2056
DOI 10.1007/s10773-010-0560-6

Solutions of a Class of Duffing Oscillators with Variable Coefficients

Pilar G. Estévez · Şengül Kuru · Javier Negro · Luis M. Nieto

Received: 20 September 2010 / Accepted: 29 October 2010 / Published online: 18 November 2010

Abstract The solutions of a class of nonlinear second-order differential equations with a cubic term in the dependent variable being related to Duffing oscillators are obtained by means of the factorization technique. The Lagrangian, the Hamiltonian and the constant of motion are also found through a correspondence with an autonomous system. A physical example is worked out in this frame.

Keywords Duffing equations · Factorization technique

1 Introduction

In the present work we will consider an interesting class of nonlinear second-order ordinary differential equations (ODE) with variable coefficients, which contains a cubic term and will be called in the sequel cubic nonlinear (CNL) equations, of the form

$$Y_{tt}(t) + F_1(t)Y_t(t) + 2F_2^2(t)Y^3(t) + F_3(t)Y(t) + F_4(t) = 0, \tag{1}$$

P.G. Estévez (✉)
Departamento de Física Fundamental, Área de Física Teórica, Universidad de Salamanca, 37008 Salamanca, Spain
e-mail: pilar@usal.es

Ş. Kuru
Department of Physics, Faculty of Science, Ankara University, 06100 Ankara, Turkey
e-mail: Sengul.Kuru@science.ankara.edu.tr

J. Negro · L.M. Nieto
Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47071 Valladolid, Spain

J. Negro
e-mail: jnegro@fta.uva.es

L.M. Nieto
e-mail: luismi@metodos.fam.cie.uva.es

where $Y(t)$ is the dependent variable, $F_i(t)$ $(i = 1, \dots, 4)$ are functions of the independent variable t, and the subindex t denotes derivative with respect to this argument, $Y_t \equiv dY/dt$. These equations can be seen as a variation of the well known Duffing oscillator

$$\theta_{tt}(t) + a\theta_t(t) + b\theta(t) + c\theta^3(t) + d\cos(\omega t) = 0, \tag{2}$$

where the coefficient a is related with the damping, b and c are determined by the restoring force, and the last term is the periodic external force. In the literature there are many studies about several equations of the Duffing type [1–8], in which different properties have been examined numerically or analytically.

In general, there are no standard techniques to solve this kind of equations (1) involving variable coefficients. However, when the coefficients are constant they have been investigated using many procedures: the elliptic averaging [9], the sinh-Gordon equation expansion [10], the homotopy-perturbation [11], and others mentioned in [9–11]. In this paper we will deal with this equation by means of the factorization method, which in general is applicable to nonlinear ordinary differential equations (ODE's) when the coefficients do not depend on the independent variable [12–15]. However, as we shall see, the case of variable coefficients can be also factorized by transforming the initial equation into a canonical form [16].

The structure of this work is the following. We start in Sect. 2 with the scale transformation of the CNL second-order differential equation (1) into its canonical form. In this way, we connect a non-autonomous system to an easier autonomous one. Then, in Sect. 3, we will find the Lagrangian and Hamiltonian for this canonical form as well as for the original differential equation, together with a relevant constant of motion. Next, in Sect. 4 the CNL second-order differential equation and its canonical form are factorized, and the solutions obtained in this way will be given in Sect. 5. We will apply our development along Sect. 6 to an interesting example belonging to this class of non-autonomous equations: a pendulum with time dependent damping and driving force. Section 7 will end this work with some remarks and conclusions.

2 Reduction to Canonical Form

As it is well known, the second-order integrable ODE's were classified by Painlevé [17]. Following the standard procedure, let us make the following scale transformation in the dependent and independent functions

$$dt = \frac{1}{\lambda(t)}dz, \qquad Y(t) = \frac{1}{\alpha(t)}W(z) \tag{3}$$

in order to reduce (1) into one of the canonical forms of Painlevé's classification. The auxiliary functions $\lambda(t)$ and $\alpha(t)$ will be found later on in terms of the initial data $F_i(t)$. Indeed, the new dependent variable $W(z)$ is required to satisfy

$$W_{zz} + 2W^3 - Q(z)W + 1 = 0, \tag{4}$$

where the function $Q(z)$ is such that $Q_{zz} = 0$. When $Q_z \neq 0$, (4) can be easily transformed into the second Painlevé transcendent [18]. Henceforth, we will concentrate on the case $Q_z = 0$, so that (4) takes the form

$$W_{zz} + 2W^3 - c_0W + 1 = 0, \tag{5}$$

where $Q(z) = c_0$ is a constant. We will see in the next sections how the solutions of (5) can be written in terms of elliptic functions. To be more explicit, applying the transformation (3) to (1) and comparing with (4), we obtain that $\lambda(t)$ and $\alpha(t)$ are defined in terms of $F_2(t)$, $F_4(t)$ as

$$\lambda = (F_2 F_4)^{1/3}, \qquad \alpha = \left(F_2^2/F_4\right)^{1/3}. \tag{6}$$

On the other hand, the functions $F_1(t)$ and $F_3(t)$ can not be independent, and they must satisfy the following relations

$$\begin{aligned} F_1 &= [\ln(F_2/F_4)]_t, \\ F_3 &= \frac{[\ln(F_2^2/F_4)]_{tt}}{3} - \frac{[\ln(F_4^2/F_2)]_t[\ln(F_2^2/F_4)]_t}{9} - c_0\,(F_2F_4)^{2/3}, \end{aligned} \tag{7}$$

where both $F_2(t)$ and $F_4(t)$ are non-vanishing functions. Therefore, we can say that (1) can be transformed into the canonical form (4) by the scale transformation (3), if and only if the functions $F_1(t)$ and $F_3(t)$ satisfy (7), respectively. Under these assumptions, the original equation (1) can be rewritten in terms of the functions $\alpha(t)$ and $\lambda(t)$, in the following way

$$Y_{tt} + \left(2\frac{\alpha_t}{\alpha} - \frac{\lambda_t}{\lambda}\right)Y_t + 2\alpha^2\lambda^2Y^3 + \left\{\frac{\alpha_{tt}}{\alpha} - \frac{\alpha_t}{\alpha}\frac{\lambda_t}{\lambda} - c_0\lambda^2\right\}Y + \frac{\lambda^2}{\alpha} = 0. \tag{8}$$

3 Lagrangians and Hamiltonians

Going back to the canonical equation (5), let us remark that it is a motion-type equation whose corresponding Lagrangian has the form

$$L_W(W, W_z, z) = \frac{1}{2}(W_z^2 - W^4 + c_0 W^2 - 2W), \tag{9}$$

being the canonical momentum $P_W = \frac{\partial L_W}{\partial W_z} = W_z$. Then, the Hamiltonian $H_W = W_z P_W - L_W$ reads

$$H_W(W, P_W, z) = \frac{1}{2}(P_W^2 + W^4 - c_0 W^2 + 2W), \tag{10}$$

where the function

$$V(W) = \frac{1}{2}(W^4 - c_0 W^2 + 2W) \tag{11}$$

can be considered as the effective potential of H_W. Since the independent variable z does not appear explicitly in (10) (it is an autonomous system), then H_W is a constant of motion, $H_W = E$, with

$$E = \frac{1}{2}\left(\left(\frac{dW}{dz}\right)^2 + W^4 - c_0 W^2 + 2W\right). \tag{12}$$

If we perform the change of variables (3) on the action

$$\int L_W(W, W_z, z)\,dz = \int L_Y(Y, Y_t, t)\,dt,$$

we obtain the Lagrangian in the initial variables Y and t:

$$L_Y(Y, Y_t, t) = \lambda(t) L_W(W, W_z, z)$$
$$= \frac{1}{2}\left(\frac{\alpha^2}{\lambda}(Y_t + [\ln\alpha]_t Y)^2 - \lambda\alpha^4 Y^4 + c_0\lambda\alpha^2 Y^2 - 2\lambda\alpha Y\right). \tag{13}$$

It can be checked that the Euler-Lagrange equation derived from this Lagrangian (13) is (8). Here, the canonical momentum $P_Y = \frac{\partial L_Y}{\partial Y_t}$ is

$$P_Y = \frac{\alpha^2}{\lambda}(Y_t + [\ln\alpha]_t Y) = \alpha P_W, \tag{14}$$

and the new Hamiltonian H_Y takes the form

$$H_Y(Y, Y_t, t) = \frac{1}{2}\left(\frac{\lambda}{\alpha^2} P_Y^2 - 2[\ln\alpha]_t P_Y Y + \lambda\alpha^4 Y^4 - c_0\lambda\alpha^2 Y^2 + 2\lambda\alpha Y\right). \tag{15}$$

Remark that H_Y is no longer a constant of motion (because it is not autonomous, due to the presence of the functions $\lambda(t)$ and $\alpha(t)$). Nevertheless, we can apply the transformation (3) to H_W, in order to get the corresponding constant of motion in the original variables:

$$\tilde{E} = \frac{1}{2}\left(\frac{P_Y^2}{\alpha^2} + \alpha^4 Y^4 - c_0\alpha^2 Y^2 + 2\alpha Y\right). \tag{16}$$

Indeed, using (15) and (16), it can be proved by a direct calculation that

$$\frac{d\tilde{E}}{dt} = \frac{\partial\tilde{E}}{\partial t} + \{\tilde{E}, H_Y\} = 0, \tag{17}$$

where $\{\cdot, \cdot\}$ are the Poisson brackets. It is well known that the damped harmonic oscillator [19] has two independent integrals of motion: one of them is energy-like and the other one is known as Bohlin's integral. Here, we want to remark that for our system it is also possible to build two similar invariants from the factorization of (12), following [12].

4 Invariants of the CNL Equations

A certain class of nonlinear second-order ODE's can be solved by a factorization technique under some conditions (see for instance [12–14, 16]). These equations frequently appear when looking for travelling wave solutions of partial nonlinear equations, such as Korteweg-deVries-Burgers [12, 13], Kadomtsev-Petviashvili [12], or Benjamin-Bona-Mahony [14]. In some cases the factorizations are directly related to first integrals of the equation.

4.1 Factorization of the Canonical Equation

In this section, we will see how we can factorize (5) in the form

$$\left(\frac{d}{dz} - L_2(W)\right)\left(\frac{d}{dz} - L_1(W)\right)W = 0. \tag{18}$$

Comparing (5) with (18), we obtain the following consistency conditions

$$L_1 + L_2 + W\frac{dL_1}{dW} = 0, \qquad L_1L_2W - 2W^3 + c_0W - 1 = 0. \tag{19}$$

These two relations are reduced to a single one

$$\frac{d(W^2L_1^2)}{dW} + 4W^3 - 2c_0W + 2 = 0, \tag{20}$$

which can be solved for L_1, and then, used to get L_2. In this way we find two solutions,

$$L_1^{\pm} = \pm\frac{\sqrt{2k_0 - W^4 + c_0W^2 - 2W}}{W}, \tag{21}$$

where k_0 is an another integration constant. Hence, from the factorization we have the following first-order ODE

$$\left(\frac{d}{dz} - L_1^{\pm}(W)\right)W = 0. \tag{22}$$

Notice that, in this case, this first-order ODE can also be obtained by the factorization of the integral of motion (12) where $k_0 = E$. Therefore, we see that when we find W from (22), we have also a solution of (5) [12–16].

4.2 Factorization of the Initial CNL Second-Order ODE

Since the coefficients of (8) depend on the independent variable t, to obtain a direct factorization is a rather difficult problem. However, we can achieve it with the help of the factorization of (5), performed in the previous subsection. Let us assume that (8) is factorized in the following form

$$\left(\frac{d}{dt} - G_2(Y,t)\right)\left(\frac{d}{dt} - G_1(Y,t)\right)Y = 0. \tag{23}$$

If we write (22) in terms of the rescaled variables, we get again a first order ODE

$$\frac{\alpha}{\lambda}\left(\frac{dY}{dt} + \left[\frac{\alpha_t}{\alpha} - \lambda L_1^{\pm}\right]Y\right) = 0, \tag{24}$$

that can be rewritten in the form

$$\left(\frac{d}{dt} - G_1^{\pm}(Y,t)\right)Y = 0 \tag{25}$$

where $G_1^{\pm}(Y,t) = -\frac{\alpha_t}{\alpha} + \lambda L_1^{\pm}$. Substituting (21) (in terms of α and Y) we obtain $G_1^{\pm}$ as

$$G_1^{\pm}(Y,t) = -\frac{\alpha_t}{\alpha} \pm \frac{\lambda}{\alpha}\frac{\sqrt{2k_0 - \alpha^4Y^4 + c_0\alpha^2Y^2 - 2\alpha Y}}{Y}. \tag{26}$$

We also check that the first order ODE (25) can be obtained from the factorization of the constant of motion $\tilde{E}$ (16).

5 Solutions of the CNL Second-Order ODE in Terms of Weierstrass Functions

As we have seen in the previous sections, the integrable forms of (1) that can be written as (8) are reducible to the canonical form (5) by means of the scale transformation (3). The solutions of (5) are obtained from the factorization (22) or, equivalently, from the integral of motion E (12) that will be rewritten as

$$\left(\frac{dW}{dz}\right)^2 = P(W), \tag{27}$$

where

$$P(W) = -W^4 + c_0 W^2 - 2W + 2E. \tag{28}$$

Let us call w_i, $i = 1, \dots, 4$ the four (in general different) roots of the polynomial (28). They satisfy the following relations [20]

$$\begin{aligned} &(w_1+w_2)(w_2+w_3)(w_1+w_3) = 2, \qquad w_1+w_2+w_3+w_4 = 0, \\ &c_0 = w_1w_2 + w_1w_3 + w_2w_3 + w_1^2 + w_2^2 + w_3^2, \qquad E = -\frac{1}{2}w_1w_2w_3w_4. \end{aligned} \tag{29}$$

Then, the solutions of (27)–(28) can be written in terms of the Weierstrass $\wp$ function as follows [21]

$$W(z) = w_i + \frac{6 - 6c_0 w_i + 12 w_i^3}{c_0 - 6w_i^2 - 12\wp(z+z_0; g_2, g_3)}, \tag{30}$$

where w_i is one of the roots of the polynomial (28). The invariants of (28) depend on c_0 and E

$$g_2 = \frac{c_0^2}{12} - 2E, \qquad g_3 = -\frac{c_0^3}{216} - \frac{c_0 E}{3} + \frac{1}{4}, \tag{31}$$

and the discriminant is $\Delta = g_2^3 - 27g_3^2$. If $\Delta = 0$, the $\wp$ function degenerates into trigonometric or hyperbolic functions [21]. When we substitute (30) into (3) we have the solutions of (8) in the form

$$Y(t) = \frac{1}{\alpha(t)}\left(w_i + \frac{6 - 6c_0 w_i + 12 w_i^3}{c_0 - 6w_i^2 - 12\wp(z(t)+z_0; g_2, g_3)}\right), \tag{32}$$

where $z(t) = \int \lambda(t)dt$. Once the constant c_0 is fixed, we have a non-symmetrical double well potential (11). Then, the real solutions of (27)–(28) have a different character according to the values of the constant E. Finally, from such solutions we can get the physical solutions of (8) by means of (32).

6 Example: A Damped and Forced Pendulum

Let us consider the equation

$$ml^2\theta_{tt} + mgl\sin\theta = -\gamma(t)\theta_t + F(t) \tag{33}$$

corresponding to a pendulum of length l under the influence of an external time-dependent force $F(t)$ and with a variable damping coefficient $\gamma(t)$. By taking the first two terms in the series development of $\sin\theta$, we get

$$\theta_{tt} + \frac{\gamma(t)}{ml^2}\theta_t - \frac{g}{6l}\theta^3 + \frac{g}{l}\theta - \frac{F(t)}{ml^2} = 0, \tag{34}$$

which can be considered as a cubic oscillator modified by damping and restoring terms. This equation is of the form (1), with the following identification of the coefficients:

$$F_1 = \frac{\gamma(t)}{ml^2}, \qquad F_2 = i\sqrt{\frac{g}{12l}}, \qquad F_3 = \frac{g}{l}, \qquad F_4 = -\frac{F(t)}{ml^2}. \tag{35}$$

Therefore, the functions (6) now take the explicit form

$$\lambda(t) = i\left(\sqrt{\frac{g}{12l}}\frac{F(t)}{ml^2}\right)^{1/3}, \qquad \alpha(t) = \left(\frac{mgl}{12F(t)}\right)^{1/3}, \tag{36}$$

so that

$$\frac{\lambda_t}{\lambda} = -\frac{\alpha_t}{\alpha} = \frac{1}{3}\frac{F_t}{F}. \tag{37}$$

According to the results of Sect. 2, (34) must be written in the form (8) in order to be integrable. Then, equating the coefficients of these equations, we have the conditions

$$F_1 = \frac{\gamma(t)}{ml^2} = -\frac{F_t}{F}, \qquad F_3 = \frac{g}{l} = -\frac{1}{3}\frac{F_{tt}}{F} + \frac{5}{9}\frac{F_t^2}{F^2} + c_0\left(\frac{gF^2}{12m^2l^5}\right)^{1/3}. \tag{38}$$

Thus, from (38) the functions $\gamma(t)$ and $F(t)$ must be related as $\gamma(t) = -ml^2 F_t/F$, and $F(t)$ must satisfy the ODE

$$\frac{g}{l} + \frac{1}{3}\frac{F_{tt}}{F} - \frac{5}{9}\frac{F_t^2}{F^2} - c_0\left(\frac{gF^2}{12m^2l^5}\right)^{1/3} = 0, \tag{39}$$

which in terms of $\alpha(t)$, given by (36), becomes

$$(\alpha^2)_{tt} + \frac{c_0 g}{6l} - \frac{2g}{l}\alpha^2 = 0. \tag{40}$$

As $\alpha(t)$ must have no vanishing points, the appropriate solution of (40) has the form

$$\alpha(t) = \sqrt{\frac{c_0}{6}}\cosh\left[\sqrt{\frac{g}{2l}}(t+t_0)\right]. \tag{41}$$

Using (41) in (36) and (38), we get the functional forms of the force and damping terms

$$F(t) = \sqrt{\frac{3}{2}}\frac{mlg}{\sqrt{c_0^3}}\left(\cosh\left[\sqrt{\frac{g}{2l}}(t+t_0)\right]\right)^{-3}, \tag{42}$$

$$\gamma(t) = ml^2\sqrt{\frac{g}{2l}}\tanh\left[\sqrt{\frac{g}{2l}}(t+t_0)\right], \tag{43}$$

which allow us to integrate the equation. Remark that $\gamma(t) > 0$ for $t > 0$ ($t_0 = 0$), therefore, we can restrict to positive time ($t > 0$), while for a negative time the formal solution is symmetric in t.

Substituting (42) in (36) we get

$$\lambda(t) = i\sqrt{\frac{g}{2lc_0}}\frac{1}{\cosh[\sqrt{\frac{g}{2l}}(t+t_0)]}. \tag{44}$$

Now, as we know the functions $\alpha(t)$ and $\lambda(t)$, we can write the scale transformations (3) in the form

$$\theta(t) = \sqrt{\frac{6}{c_0}}\frac{1}{\cosh[\sqrt{\frac{g}{2l}}(t+t_0)]}W(z), \tag{45}$$

$$dt = -i\sqrt{\frac{2lc_0}{g}}\cosh\left[\sqrt{\frac{g}{2l}}(t+t_0)\right]dz. \tag{46}$$

Here, it is useful to express z in terms of t:

$$z = i\frac{2}{\sqrt{c_0}}\arctan\left[\tanh\left(\frac{1}{2}\sqrt{\frac{g}{2l}}(t+t_0)\right)\right]. \tag{47}$$

Therefore, we see from (47) that when the range of t is $(-\infty, +\infty)$, then z varies in a finite imaginary interval:

$$-\frac{\pi}{\sqrt{c_0}} < \operatorname{Im} z < \frac{\pi}{\sqrt{c_0}}. \tag{48}$$

From (47) we see that since $t \in \mathbf{R}$, z must be purely imaginary. For this reason, in this example it is convenient to express the canonical equation (5) in terms of the variable $x = -iz$. In particular, the invariant $\overline{E} = -E$ is

$$\overline{E} = \frac{1}{2}\left(\left(\frac{d\overline{W}}{dx}\right)^2 - \overline{W}^4 + c_0\overline{W}^2 - 2\overline{W}\right), \tag{49}$$

where $\overline{W}(x) = W(ix)$. This equation corresponds to a system with an inverted double well potential (see Fig. 1)

$$\overline{V}(\overline{W}) = \frac{1}{2}(-\overline{W}^4 + c_0\overline{W}^2 - 2\overline{W}). \tag{50}$$

Now, we can make a correspondence of the physical solutions of the canonical function $\overline{W}(x)$ and the solutions of our example $\theta(t)$. We will consider here only the bounded motions of $\overline{W}(x)$ that also go into bounded motions of $\theta(t)$. These are characterized by the values of the constant of motion satisfying $\overline{E}_3 \le \overline{E} \le \overline{E}_2$. The resulting motions $\theta(t)$ have been plotted in Figs. 2 and 3. The solution $\overline{W}(x)$ for $\overline{E} = \overline{E}_3$ is constant, we have a stable equilibrium point, so the motion $\theta(t)$ in this case has a similar shape as shown in Fig. 2 (left) corresponding to the unstable equilibrium point at $\overline{E} = \overline{E}_2$. We have also a critical motion at $\overline{E} = \overline{E}_2$ described by a degenerate $\wp$ function shown in Fig. 2 (right). For $\overline{E}_3 < \overline{E} < \overline{E}_2$ the motion is bounded and overdamped as it is shown in Fig. 3. Remark that for $\overline{E} > \overline{E}_2$ the solutions $\overline{W}(x)$ are non-bounded, due to the motion under a potential barrier (Fig. 1).

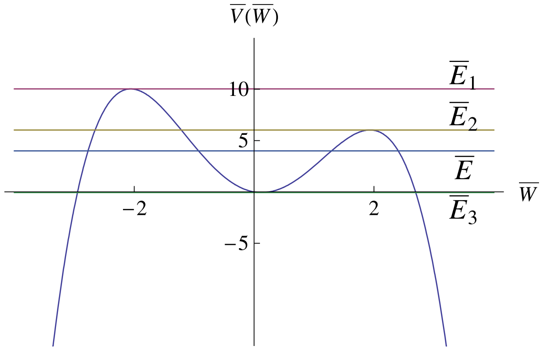

Fig. 1 Plot of the potential $\overline{V}(\overline{W})$ and the values of the energy, $\overline{E}_1, \overline{E}_2, \overline{E}_3$, corresponding to the three double roots of the polynomial. One of the selected energies $\overline{E}$ is also plotted. We have taken $c_0 = 8$

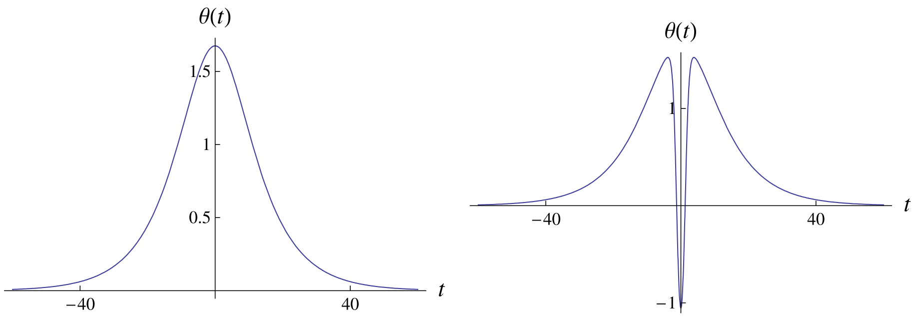

Fig. 2 Plot of the solutions $\theta(t)$ for the constants of motion $\overline{E}_2 = 6.0$. The *left figure* corresponds to an equilibrium point of $\overline{W}(x) = \text{const.}$, while the *right one* is for a hyperbolic-type motion of $\overline{W}(x)$. We have taken $c_0 = 8$, $l = 50$, $m = 1$, $g = 1$

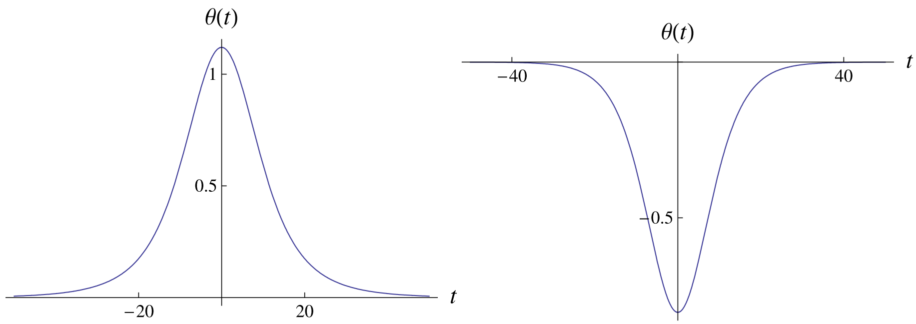

Fig. 3 Plot of the solutions $\theta(t)$ for the constants of motion $\overline{E} = 4.0$. The *left figure* corresponds to the motion starting from the minimun value of $\overline{W}(x)$ at $x = 0$, while the *right one* starts in the maximun of $\overline{W}(x)$ at $x = 0$. We have taken $c_0 = 8$, $l = 50$, $m = 1$, $g = 1$

7 Conclusions

In this paper we have factorized both a class of general nonlinear second-order ODE with variable coefficients including a cubic term, and the transformed canonical equation which is a second-order ODE with constant coefficients. We have shown that the reduction to canonical form imposes the restrictions (7) on the variable coefficients of the initial equation. The corresponding Lagrangians and Hamiltonians have been found, leading to a constant of motion which in this case is directly related to the factorization. The solutions obtained from the factorization have been expressed in terms of the Weierstrass $\wp$ function. For special values of the constant of motion E and the parameter c_0, some particular solutions can be expressed in terms of trigonometric and hyperbolic functions. Finally, we have shown an example, together with some explicit solutions, where this kind of equations appear in a physical context. The allowed time dependent coefficients of this example correspond to a kick force (42) applied at time $t = t_0$ and damping term (43) having a constant asymptotic value at $t \to +\infty$. We have described the behaviour of the motion for a critical energy $\overline{E}_2$ in Fig. 2 and for an overdamping oscillation in Fig. 3.

Acknowledgements This work has been partially supported by the Spanish MEC (Projects MTM2009-10751, FIS2009-09002) and Junta de Castilla y León (Excellence Project GR224). ŞK acknowledges warm hospitality at the Department of Theoretical Physics of the University of Valladolid, where part of this work has been carried out.

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