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2016 Eur. J. Phys. 37 055401

(http://iopscience.iop.org/0143-0807/37/5/055401)

IOP Publishing European Journal of Physics
Eur. J. Phys. **37** (2016) 055401 (16pp) doi:10.1088/0143-0807/37/5/055401

The one dimensional Schrödinger equation: symmetries, solutions and Feynman propagators

Jose M Cerveró and Pablo P Polo

Departamento de Física Fundamental, Facultad de Ciencias, Universidad de Salamanca, E-37008 Salamanca, Spain

E-mail: cervero@usal.es and ppajuelo94@usal.es

Received 6 April 2016, revised 13 May 2016
Accepted for publication 20 May 2016
Published 16 June 2016

Abstract

A simple method to find the symmetries of the Schrödinger equation in one dimension with arbitrary potentials is presented. The method hereby used can be of interest to students in quantum mechanics at the undergraduate level. Several physical questions arising from this symmetry analysis are also discussed. Finally, some attention is paid to how the result leads to the explicit description of the exact propagators in the linear and quadratic case without using the more popular method of describing the classical action in the stationary phase approximation.

Keywords: quantum mechanics, elementary symmetries, path integrals

1. Introduction

More than 40 years ago Niederer and Barut [1, 2, 3–5] found in a set of seminal papers the symmetries of the Schrödinger equation, using sophisticated methods of Lie group theory. This result is of primary importance for studying the structure of the solutions of the Schrödinger equation, and has been used several times to uncover surprising features of some solutions initially identified as 'anomalous' but actually well understood using Niederer's transformations [15, 20, 21]. Despite the time elapsed between the last two references, it is surprising that the wealth of results that the symmetry approach provides has passed almost unnoticed from the practical point of view. Actually, all these results on the symmetry approach are based in the well known but not always widely used Lie theory of symmetries for differential equations, either ordinary (ODE) or partial (PDE). A complete study of Lie theory can be found in [6]. As this is a highly non trivial book for the undergraduate level one can find more accessible references—already applied to operators in quantum mechanics—in [7] and [8].

A modest aim of the present contribution is to fill this gap by simplifying Niederer's method, making it more accessible to teachers and students in undergraduate programs and discussing the physics encrypted within the symmetric approach. In particular, one can show that not only are transformations of solutions derived from it, but more advanced results can also be obtained, such as exact path integrals in particular cases and other hidden features of interest, as we shall see in the different sections of the paper, which is organized as follows. In section 2 we concentrate on the symmetries of the one dimensional Schrödinger equation. In sections 3 and 4 the properties of the symmetries will be discussed and the symmetries assigned to specific physical properties. The link between symmetries and path integrals is treated in section 5. A brief appendix shows that the calculations are embarrassingly simple.

2. Symmetry transformations of the one dimensional Schrödinger equation

Suppose that we write down the one dimensional Schrödinger equation living in a manifold $\mathcal{M}_1\{x_1, t_1\}$ in the usual form

$$\left\{\mathrm{i}\hbar\frac{\partial}{\partial t_1} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - V_1(x_1, t_1)\right\}\Psi_1(x_1, t_1) = 0. \tag{1}$$

Instead of going through all the geometric formalism of Lie group theory applied to PDEs [6], we take instead a shortcut by restricting ourselves to the so called linear rigid transformations or scale transformations [9]

$$x_2 = f(t_1)x_1 \quad + \quad h(t_1)x_0 \tag{2a}$$

$$\mathrm{d}t_2 = f^2(t_1)\mathrm{d}t_1 \tag{2b}$$

which transform the variables in $\mathcal{M}_1\{x_1, t_1\}$ to variables in $\mathcal{M}_2\{x_2, t_2\}$ with a Schrödinger equation of the form

$$\left\{\mathrm{i}\hbar\frac{\partial}{\partial t_2} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} - V_2(x_2, t_2)\right\}\Psi_2(x_2, t_2) = 0. \tag{3}$$

One also has to remember that the transformation of the wave functions $\Psi(x_1, t_1)$ and $\Psi(x_2, t_2)$ is non trivial as, according to quantum mechanics, a phase must be allowed to appear in the transformation due to the projective nature of the wave function that defines the state up to an arbitrary phase

$$\Psi_2(x_2, t_2) = \exp\{\mathrm{i}\Theta(x_1, t_1)\}\Psi_1(x_1, t_1). \tag{4}$$

The transformation lead us to the following conclusions. Firstly, the phase must be transformed in the following form

$$\Theta(x_1, t_1) = \frac{m}{\hbar}\left\{\frac{\dot{f}(t_1)}{2f(t_1)}x_1^2 + \frac{\dot{h}(t_1)}{f(t_1)}x_1x_0 + \lambda(t_1)\right\} \tag{5}$$

and the potentials transform one into the other according to the rule

$$\begin{aligned} V_2(x_2, t_2) = \frac{1}{f^2(t_1)}\Bigg[& V_1(x_1, t_1) - \frac{m}{2f^2(t_1)}\{f(t_1)\ddot{f}(t_1) - 2\dot{f}^2(t_1)\}x_1^2 \\ & - \frac{mx_0}{f^2(t_1)}\{f(t_1)\ddot{h}(t_1) - 2\dot{f}(t_1)\dot{h}(t_1)\}x_1 + \frac{mx_0^2}{2f^2(t_1)}\dot{h}^2(t_1) + \mathrm{i}\hbar\frac{\dot{f}(t_1)}{2f(t_1)} - m\dot{\lambda}(t_1)\Bigg] \end{aligned} \tag{6}$$

where $\lambda(t_1)$ is an arbitrary function of the time t_1 and

$$\dot{f}(t_1) = \frac{\mathrm{d}f}{\mathrm{d}t_1}; \qquad \dot{h}(t_1) = \frac{\mathrm{d}h}{\mathrm{d}t_1} \tag{7a}$$

$$\ddot{f}(t_1) = \frac{\mathrm{d}^2 f}{\mathrm{d}t_1^2}; \qquad \ddot{h}(t_1) = \frac{\mathrm{d}^2 h}{\mathrm{d}t_1^2} \qquad \text{with} \qquad \dot{\lambda}(t_1) = \frac{\mathrm{d}\lambda}{\mathrm{d}t_1}. \tag{7b}$$

The main calculations leading to these formulae are collected in the final part of the paper as an appendix.

3. Invariance and group transformations

Let us first assume that we are dealing with the case of invariance of the free particle. Hence $V_1(x_1, t_1) = V_2(x_2, t_2) = 0$. In the notation of (2a) and (2b)

$$\begin{aligned} x_2 &= f(t_1)x_1 + h(t_1)x_0 \\ \mathrm{d}t_2 &= f^2(t_1)\mathrm{d}t_1 \end{aligned}.$$

The following set of functions make the rest of the parentheses in (6) vanish

1.

$$f(t_1) = 1; \quad x_2 = x_1 + x_0 \tag{9a}$$

$$h(t_1) = 1; \quad t_2 = t_1 \tag{9b}$$

where x_0 is an arbitrary constant and $\lambda(t_1) = 0$.

2.

$$f(t_1) = 1; \quad x_2 = x_1 \tag{10a}$$

$$h(t_1) = 0; \quad t_2 = t_1 + t_0 \tag{10b}$$

where t_0 is an arbitrary constant and $\lambda(t_1) = 0$.

3.

$$f(t_1) = 1; \quad x_2 = x_1 + v_0 t_1 \tag{11a}$$

$$h(t_1) = \frac{v_0}{x_0}t_1; \quad t_2 = t_1 \tag{11b}$$

where v_0 is an arbitrary constant. Furthermore $\lambda(t_1)$ takes the following non trivial value

$$\lambda(t_1) = \frac{v_0^2 t_1}{2}.$$

4.

$$f(t_1) = \exp\{\delta\}; \quad x_2 = \exp\{\delta\}x_1 \tag{12a}$$

$$h(t_1) = 0; \quad t_2 = \exp\{2\delta\}t_1 \tag{12b}$$

where δ is an arbitrary constant and $\lambda(t_1) = 0$.

5.

$$f(t_1) = \frac{1}{1 + \omega t_1}; \quad x_2 = \frac{x_1}{1 + \omega t_1} \tag{13a}$$

$$h(t_1) = 0; \qquad t_2 = \frac{t_1}{1 + \omega t_1} \tag{13b}$$

where ω is an arbitrary constant. In this case, $\lambda(t_1)$ is forced to take the value

$$\lambda(t_1) = -\frac{i\hbar}{2m} \ln(1 + \omega t_1).$$

Note that this implies always that $(t_1 - t_0) > 0$ as it corresponds to the argument of a logarithm which has to be always positive. As the above expression appears in various places in the rest of the paper one has to take into account this restriction in dealing with single-valuedness and analyticity of the functions that contain this temporal interval. The transformations 1, 2 and 3 are easily recognized as the kinematical symmetries corresponding to space translation, time translation and Galilean transformation. As we can see, the symmetry approach surprises us with two more unexpected symmetries (i.e cases 4 and 5) which have been known for long time as the Niederer transformations. They correspond to constant and non constant inhomogeneous space-time transformations, which will be very useful for the future analysis of the solutions.

The transformations 1, 2 and 3 are also symmetries which leave unchanged the entire one dimensional Schrödinger equation, such that for them

$$V_1(x_1, t_1) = V_2(x_2, t_2)$$

as can easily be checked. However, for the Galilean transformation 3, the phase does change in a non trivial way. Using the form of the functions given in (11*a*) and (11*b*)

$$f(t_1) = 1; \qquad h(t_1) = \frac{v_0}{x_0} t_1 \qquad \text{and} \qquad \dot{\lambda}(t_1) = \frac{v_0^2}{2} \tag{14}$$

the phase takes the form

$$\Theta(x_1, t_1) = \frac{m}{\hbar}\left\{\frac{\dot{f}(t_1)}{2f(t_1)} x_1^2 + \frac{\dot{h}(t_1)}{f(t_1)} x_1 x_0 + \lambda(t_1)\right\} = \frac{m}{\hbar}\left\{v_0 x_1 + \frac{1}{2} v_0^2 t_1\right\} \tag{15}$$

and the wave function

$$\Psi_2(x_2, t_2) = \exp\left\{\frac{im}{\hbar}\left(v_0 x_1 + \frac{1}{2} v_0^2 t_1\right)\right\}\Psi_1(x_1, t_1). \tag{16}$$

This non trivial phase factor has to be taken into account for any Galilean transformation in quantum mechanics. Otherwise energy and momentum conservation do not work as they should [11, 12]. We analyze this apparent paradox in the next section.

4. Galilei group: paradoxes and structure

In the previous section, we have analyzed the importance of the phase in the invariance group of the free Schrödinger equation. Now we turn our attention to the application of these concepts to both the structure of the Galilei group in one spatial dimension as well as clarifying the meaning of the conservation laws of energy and momentum that must also hold in the quantum domain. This last point of energy and momentum conservation has not always been correctly understood, giving rise to the so-called pseudoparadoxes, mainly attributed to Landé [10]. We shall explain first the core of the pseudoparadox, solved by Levy-Léblond in two illuminating contributions [11, 12]. To give to the reader a clear perspective we shall first deal with the apparent problem of non invariance of the Schrödinger equation under Galilei transformations. The classical laws of energy and momentum conservationn under a Galilei transformation are

$$p' = p + mv_0 \tag{17}$$

$$E' = E + pv_0 + \frac{1}{2}mv_0^2. \tag{18}$$

The first law is trivial but the second one is less known. Actually, it can be better understood if we consider the potential as the internal energy of the system. Therefore this quantity is obviously invariant under the transformation and one can write

$$E - \frac{p^2}{2m} = E' - \frac{p'^2}{2m} \Longrightarrow E' = E - \frac{p^2}{2m} + \frac{1}{2m}(p + mv_0)^2 = E + pv_0 + \frac{1}{2}mv_0^2$$

as above. Now consider the simplest wave function satisfying the Schrödinger equation. It is a plane wave of the form

$$\Psi(x, t) = \exp\left\{2\pi \mathrm{i}\left(\frac{x}{\lambda} - \nu t\right)\right\} \quad \text{where} \quad \frac{2\pi}{\lambda} = k \quad \text{and} \quad 2\pi\nu = \omega.$$

If a pure Galilei transformation is now performed on this solution one obtains

$$\Psi(x, t) = \exp\left\{2\pi \mathrm{i}\left(\frac{x}{\lambda} - \nu t\right)\right\} = \exp\left\{2\pi \mathrm{i}\left(\frac{x' - v_0 t)}{\lambda} - \nu t'\right)\right\}$$
$$= \exp\left\{2\pi \mathrm{i}\left(\frac{x'}{\lambda} - \left(\nu + \frac{v_0}{\lambda}\right)t\right)\right\}.$$

So, the conservation laws, after an overall multiplication for the Planck constant h, yields after comparison with the initial wave function

$$\left(\frac{h}{\lambda'} = \frac{h}{\lambda}\right) \Longrightarrow p' = p; \qquad \left(h\nu' = h\nu + \frac{hv_0}{\lambda}\right) \Longrightarrow E' = E + pv_0.$$

Both relationships are in sharp disagreement with the laws of energy and momentum conservation (17)–(18). However, if one includes the phase appearing in (16) as a consequence of the performed Galilean transformation, then we have

$$\Psi(x', t') = \exp\left\{\frac{\mathrm{i}m}{\hbar}\left(v_0 x + \frac{1}{2}v_0^2 t\right)\right\}. \tag{19}$$

For the previous plane wave solution we obtain

$$\Psi(x', t') = \exp\left\{\frac{\mathrm{i}m}{\hbar}\left(v_0 x + \frac{1}{2}v_0^2 t\right)\right\}\exp\left\{2\pi \mathrm{i}\left(\frac{x}{\lambda} - \nu t\right)\right\}. \tag{20}$$

Taking into account (11*a*)–(11*b*) and after some trivial arithmetic operations

$$\Psi(x', t') = \exp\left\{2\pi \mathrm{i}\left(\left[\frac{1}{\lambda} + \frac{mv_0}{h}\right]x' - \left[\nu + \frac{v_0}{\lambda} + \frac{mv_0^2}{2h}\right]t'\right)\right\}. \tag{21}$$

So, we have

$$\frac{1}{\lambda'} = \frac{1}{\lambda} + \frac{mv_0}{h} \tag{22}$$

$$\nu' = \nu + \frac{v_0}{\lambda} + \frac{mv_0^2}{2h} \tag{23}$$

and multiplying by h one finally obtains

$$p' = p + mv_0 \tag{24}$$

$$E' = E + pv_0 + \frac{1}{2}mv_0^2. \tag{25}$$

This is in complete agreement with (17)–(18). This trivial exercise shows us the importance of including the correspondent phases in the wave functions whichever transformation is performed on the Schrödinger equation, and very particularly for those affecting the invariance under crucial physical laws as has been the case for the pure Galilean transformations. Now we turn to the quantum mechanical group structure of the set of Galilean transformations in one spatial dimension. In quantum mechanics a given symmetry is always associated with the existence of a projective unitary operator acting on the quantum state [13]. Therefore in a space-time governed by Galilean laws, a set of such operators should also be defined. These three finite transformations laws are

$$\begin{aligned} x' &= x + x_0 \\ t' &= t + t_0 \\ x' &= x + v_0 t. \end{aligned}$$

Let us now define the unitary operators corresponding to these kinematical transformations

$$\hat{\mathbf{U}}(x_0)|x\rangle = \exp\left\{-\frac{\mathrm{i}}{\hbar}x_0\hat{\mathbf{P}}\right\}|x\rangle = |x + x_0\rangle \tag{26}$$

$$\hat{\mathbf{U}}(p_0)|x\rangle = \exp\left\{-\frac{\mathrm{i}}{\hbar}p_0\hat{\mathbf{X}}\right\}|x\rangle = \exp\{-\mathrm{i}k_0x\}|x\rangle \tag{27}$$

$$\hat{\mathbf{U}}(x_0)|p\rangle = \exp\left\{-\frac{\mathrm{i}}{\hbar}x_0\hat{\mathbf{P}}\right\}|p\rangle = \exp\{-\mathrm{i}kx_0\}|p\rangle \tag{28}$$

$$\hat{\mathbf{U}}(p_0)|p\rangle = \exp\left\{-\frac{\mathrm{i}}{\hbar}p_0\hat{\mathbf{X}}\right\}|p\rangle = |p - p_0\rangle. \tag{29}$$

We shall prove in what follows that $\hat{\mathbf{U}}(x_0)$ acting on the state $|x\rangle$ yields the new state $|x + x_0\rangle$. Let us expand the operator $\hat{\mathbf{U}}(x_0)$ in series of x_0 until the first order of such a parameter x_0. One easily obtains

$$\exp\left\{-\frac{\mathrm{i}}{\hbar}x_0\hat{\mathbf{P}}\right\} = \left\{\hat{\mathbf{1}} - \frac{\mathrm{i}}{\hbar}x_0\hat{\mathbf{P}} + \mathcal{O}(x_0^2)\right\}. \tag{30}$$

Now we apply the expansion on an eigenstate of the position operator $\hat{\mathbf{X}}$. The operation gives as a result another eigenstate of $\hat{\mathbf{X}}$, with eigenvalue $x + x_0$

$$\left\{\hat{\mathbf{1}} - \frac{\mathrm{i}}{\hbar}x_0\hat{\mathbf{P}} + \mathcal{O}(x_0^2)\right\}|x\rangle = |x + x_0\rangle. \tag{31}$$

To show this, we apply $\hat{\mathbf{X}}$ to both members of (31)

$$\hat{\mathbf{X}}\left\{\hat{\mathbf{1}} - \frac{\mathrm{i}}{\hbar}x_0\hat{\mathbf{P}} + \mathcal{O}(x_0^2)\right\}|x\rangle = \hat{\mathbf{X}}|x + x_0\rangle. \tag{32}$$

In the left-hand side one can apply the Heisenberg canonical commutation relation $[\hat{\mathbf{X}}, \hat{\mathbf{P}}] = \mathrm{i}\hbar\hat{\mathbf{1}} \Longrightarrow \hat{\mathbf{X}}\hat{\mathbf{P}} = \mathrm{i}\hbar\hat{\mathbf{1}} + \hat{\mathbf{P}}\hat{\mathbf{X}}$, yielding

$$\left\{x + x_0 - \frac{\mathrm{i}}{\hbar}xx_0\hat{\mathbf{P}} + \mathcal{O}(x_0^2)\right\}|x\rangle. \tag{33}$$

In the right-hand side of (32), we operate using (26) as

$$\hat{\mathbf{X}}|x + x_0\rangle = (x + x_0)|x + x_0\rangle = (x + x_0)\left\{\hat{\mathbf{1}} - \frac{\mathrm{i}}{\hbar}x_0\hat{\mathbf{P}} + \mathcal{O}(x_0^2)\right\}|x\rangle$$
$$= \left\{x + x_0 - \frac{\mathrm{i}}{\hbar}xx_0\hat{\mathbf{P}} - \frac{\mathrm{i}}{\hbar}x_0^2\hat{\mathbf{P}} + \mathcal{O}(x_0^2)\right\}|x\rangle = \left\{x + x_0 - \frac{\mathrm{i}}{\hbar}xx_0\hat{\mathbf{P}} + \mathcal{O}(x_0^2)\right\}|x\rangle. \tag{34}$$

This is of the form given by (33). So, we have locally proven that $\hat{\mathbf{U}}(x_0)$ on the state $|x\rangle$ yields the new state $|x + x_0\rangle$. Identical reasoning must be followed to prove the remaining equations (27)–(29). Recall that as always $p = \hbar k$ y $p_0 = \hbar k_0$. The pure Galilean transformation must be a combination of the operators $\hat{\mathbf{X}}$ and $\hat{\mathbf{P}}$, acting in the form

$$\hat{\mathbf{U}}(v_0)|x\rangle = \exp\left\{-\frac{\mathrm{i}}{\hbar}(v_0)\hat{\mathbf{G}}\right\}|x\rangle \Longrightarrow |x + v_0 t\rangle \tag{35}$$

$$\hat{\mathbf{U}}(v_0)|p\rangle = \exp\left\{-\frac{\mathrm{i}}{\hbar}(v_0)\hat{\mathbf{G}}\right\}|p\rangle \Longrightarrow |p + mv_0\rangle \tag{36}$$

because according to the kinematical laws of the pure Galilean transformation, the following classical relationships must hold

$$p' = p + mv_0 \tag{37}$$

$$E' = E + pv_0 + \frac{1}{2}mv_0^2. \tag{38}$$

This linear combination is of the form

$$\hat{\mathbf{G}} = t\hat{\mathbf{P}} - m\hat{\mathbf{X}}. \tag{39}$$

One can show that, besides the well known Heisenberg canonical commutation relation

$$[\hat{\mathbf{X}}, \hat{\mathbf{P}}] = \mathrm{i}\hbar\hat{\mathbf{1}} \tag{40}$$

the following commutation relations hold

$$[\hat{\mathbf{P}}, \hat{\mathbf{G}}] = \mathrm{i}\hbar m\hat{\mathbf{1}}; \quad [\hat{\mathbf{H}}, \hat{\mathbf{P}}] = 0; \quad [\hat{\mathbf{H}}, \hat{\mathbf{G}}] = \mathrm{i}\hat{\mathbf{P}} \tag{41}$$

where $\hat{\mathbf{H}}$ is the generator of time translations. These three commutation relations constitute the quantum Galilei group in one spatial dimension. Finally we calculate the action of $\hat{\mathbf{U}}(v_0)$ on the states $|x\rangle$ and $|p\rangle$. The unitary operator $\hat{\mathbf{U}}(v_0)$ can be written in two different but equivalent forms

$$\hat{\mathbf{U}}(v_0) = \exp\left\{-\frac{\mathrm{i}}{\hbar}v_0\hat{\mathbf{G}}\right\} = \exp\left\{-\frac{\mathrm{i}mv_0^2}{2\hbar}t\right\}\exp\left\{\frac{\mathrm{i}mv_0}{\hbar}\hat{\mathbf{X}}\right\}\exp\left\{-\frac{\mathrm{i}v_0 t}{\hbar}\hat{\mathbf{P}}\right\}$$
$$= \exp\left\{+\frac{\mathrm{i}mv_0^2}{2\hbar}t\right\}\exp\left\{-\frac{\mathrm{i}v_0 t}{\hbar}\hat{\mathbf{P}}\right\}\exp\left\{\frac{\mathrm{i}mv_0}{\hbar}\hat{\mathbf{X}}\right\} \tag{42}$$

where we have used a particular case of the Baker–Campbell–Haussdorf formula

$$\exp\{\hat{\mathbf{A}} + \hat{\mathbf{B}}\} = \exp\{\hat{\mathbf{A}}\}\exp\{\hat{\mathbf{B}}\}\exp\left\{-\frac{1}{2}[\hat{\mathbf{A}}, \hat{\mathbf{B}}]\right\} \tag{43}$$

that is correct if and only if

$$[\hat{\mathbf{A}}, [\hat{\mathbf{A}}, \hat{\mathbf{B}}]] = [\hat{\mathbf{B}}, [\hat{\mathbf{A}}, \hat{\mathbf{B}}]] = 0. \tag{44}$$

The next step is to apply the operator on an eigenstate of $\hat{\mathbf{X}}$, in the two forms shown in (42)

$$\hat{\mathbf{U}}(v_0)|x\rangle = \exp\left\{\frac{\mathrm{i}mv_0^2}{2\hbar}t\right\}\exp\left\{-\frac{\mathrm{i}}{\hbar}(v_0 t)\hat{\mathbf{P}}\right\}\exp\left\{\frac{\mathrm{i}mv_0}{\hbar}\hat{\mathbf{X}}\right\}|x\rangle$$

and using (26)–(29) we finally obtain

$$\begin{aligned}\hat{\mathbf{U}}(v_0)|x\rangle &= \exp\left\{\frac{\mathrm{i}mv_0^2}{2\hbar}t\right\}\exp\left\{\frac{\mathrm{i}mv_0 x}{\hbar}\right\}\exp\left\{-\frac{\mathrm{i}}{\hbar}(v_0 t)\hat{\mathbf{P}}\right\}|x\rangle \\ &= \exp\left\{\frac{\mathrm{i}m}{\hbar}(v_0 x + \frac{1}{2}v_0^2 t)\right\}|x + v_0 t\rangle.\end{aligned} \tag{45}$$

Likewise one can obtain the action of the factorized operator on an eigenstate of $\hat{\mathbf{P}}$

$$\hat{\mathbf{U}}(v_0)|p\rangle = \exp\left\{-\frac{\mathrm{i}mv_0^2}{2\hbar}t\right\}\exp\left\{\frac{\mathrm{i}mv_0}{\hbar}\hat{\mathbf{X}}\right\}\exp\left\{-\frac{\mathrm{i}}{\hbar}(v_0 t)\hat{\mathbf{P}}\right\}|p\rangle.$$

Thus, we obtain

$$\begin{aligned}\hat{\mathbf{U}}(v_0)|p\rangle &= \exp\left\{-\frac{\mathrm{i}mv_0^2}{2\hbar}t\right\}\exp\left\{-\frac{\mathrm{i}pv_0}{\hbar}t\right\}\exp\left\{\frac{\mathrm{i}mv_0}{\hbar}\hat{\mathbf{X}}\right\}|p\rangle \\ &= \exp\left\{-\frac{\mathrm{i}t}{\hbar}\left(v_0 p + \frac{1}{2}mv_0^2\right)\right\}|p + mv_0\rangle.\end{aligned} \tag{46}$$

So, as was announced previously, the actions on eigenstates of $\hat{\mathbf{X}}$ and $\hat{\mathbf{P}}$ of the operator $\hat{\mathbf{U}}(v_0)$ are

$$\hat{\mathbf{U}}(v_0)|x\rangle = \exp\left\{\frac{\mathrm{i}m}{\hbar}\left(v_0 x + \frac{1}{2}v_0^2 t\right)\right\}|x + v_0 t\rangle \tag{47}$$

$$\hat{\mathbf{U}}(v_0)|p\rangle = \exp\left\{-\frac{\mathrm{i}t}{\hbar}\left(v_0 p + \frac{1}{2}mv_0^2\right)\right\}|p + mv_0\rangle \tag{48}$$

where the non trivial phases generated on the quantum states of position and momentum $|x\rangle$ and $|p\rangle$ by the unitary operator of pure Galilean transformation $\hat{\mathbf{U}}(v_0)$ clearly appear. This result shows in the framework of the mathematical formalism of quantum mechanics the observationns that were made using only wave mechanics in the first part of this section. Now we can repeat the wave mechanical calculation using the more rigorous formalism of the Dirac bra-ket notation. Let us first choose a wave function of a free particle in the usual form

$$\boldsymbol{\Psi}(x, t) = \mathbf{A}\exp\{\mathrm{i}(kx + \omega t)\} = \mathbf{A}\exp\left\{\frac{\mathrm{i}}{\hbar}(px + Et)\right\} \tag{49}$$

where we have used the De Broglie and Planck momentum and energy relationships: $p = \hbar k$ and $E = \hbar\omega$. Then we change the variables in both phase and argument

$$\boldsymbol{\Psi}(x,\, t) \to \hat{\mathbf{U}}(v_0) \to \mathbf{A}\exp\left\{\frac{\mathrm{i}m}{\hbar}\left(v_0 x + \frac{1}{2}v_0^2 t\right)\right\}\exp\left\{\frac{\mathrm{i}}{\hbar}(p(x + v_0 t) + Et)\right\} \tag{50}$$

and the final state reads as it should as

$$\boldsymbol{\Psi}(x' = x + v_0 t,\, t' = t) = \mathbf{A}\exp\left\{\frac{\mathrm{i}}{\hbar}\left\{(p + mv_0)x + \left(E + pv_0 + \frac{1}{2}mv_0^2\right)t\right\}\right\} \tag{51}$$

in full agreement with (37) and (38).

To end this section we would like to mention that a complete study of the Galilean Group in (2 + 1) dimensions has been carried out by S K Bose in reference [14] where some topological differences are shown due to the different nature of the space-time in which his analysis is carried out.

5. Eliminating potentials

Equation (6) clearly shows at most quadratic dependence in x_1. This is interesting because the quadratic (harmonic oscillator) and linear potentials have been known for a long time to be equivalent to the free particle under appropriate transformations. Here, expression (6) is reminiscent of this property. In fact if the initial potential $V_1(x_1,\, t_1)$ takes the form either of a linear potential or that of the harmonic oscillator, it can be transformed in the Schrödinger equation for the free particle ($V_2(x_2,\, t_2) = 0$) by eliminating them from (6) by means of adequate choices of the respective functions $f(t_1)$, $h(t_1)$ and $\lambda(t_1)$ in each case. To find such a set of functions in both cases will be the subject of this section.

Case 1. The linear potential

Let us consider the following linear rigid transformation

$$x_2 = x_1 + \frac{1}{2}gt_1^2 \tag{51a}$$

$$t_2 = t_1. \tag{51b}$$

Obviously in this case

$$f(t_1) = 1; \qquad h(t_1) = \frac{g}{2x_0}t_1^2; \qquad \dot{\lambda}(t_1) = \frac{1}{2}g^2 t_1^2. \tag{52}$$

With this choice equation (6) for the case of a free particle yields

$$V_2(x_2,\, t_2) = V_1(x_1,\, t_1) - mgx_1 = 0$$

and the phase $\Theta(x_1,\, t_1)$ in equation (5) yields

$$\Theta(x_1,\, t_1) = \frac{m}{\hbar}\left\{gx_1 t_1 + \frac{1}{6}g^2 t_1^3\right\}.$$

Therefore the wave functions transform according to the law

$$\Psi_2(x_2,\, t_2) = \exp\left\{\frac{img}{\hbar}\left(x_1 t_1 + \frac{1}{6}gt_1^3\right)\right\}\Psi_1(x_1,\, t_1).$$

For example, dropping all subindices, the free Schrödinger equation reads

$$\left\{\mathrm{i}\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right\}\Psi(x,\, t) = 0$$

where here the fundamental length of this system is defined as

$$l_0 = \left(\frac{\hbar^2}{2m^2 g}\right)^{\frac{1}{3}}$$

and g is a constant with the dimensions of an acceleration. The solution of the free Schrödinger equation reads

$$\Psi(x,\, t) = l_0^{-\frac{1}{2}} \exp\left\{\mathrm{i}\frac{mgt}{\hbar}\left(x - \frac{1}{3}gt^2\right)\right\} \mathbf{Ai}\left\{\frac{1}{l_0}\left(x - \frac{1}{2}gt^2\right)\right\}.$$

As the solution of the linear potentials is given in terms of Airy functions, this is the accelerated wave packet discovered by Berry and Balasz [15] as a solution of the free Schrödinger equation. Also looking at our transform we can give a simple physical meaning to this solution as the reflection of the equivalence principle in quantum mechanics: accelerated free motion is equivalent to the motion of a free particle in a homogeneous and constant gravitational field. The transformation generates a non trivial phase that has recently been measured using neutron interferometry. These theoretical considerations have also been subjected to experimental checking very recently. See the ingenious but extremely precise and successful experiments in references [16] and [17].

Case 2. The harmonic oscillator

Let us now consider the following linear rigid transformation [3]

$$x_2 = \frac{x_1}{\cos \omega_0 t_1} \tag{53a}$$

$$t_2 = \frac{1}{\omega_0}\tan \omega_0 t_1 \Longrightarrow \mathrm{d}t_1 = \frac{\mathrm{d}t_2}{(1 + \omega_0^2 t_2^2)}. \tag{53b}$$

The mapping transforms harmonic motion in the interval $\left\{-\frac{\pi}{2\omega_0},\, +\frac{\pi}{2\omega_0}\right\}$ (half of the period) into the free dynamics for $-\infty < t < +\infty$. These restrictions on the intervals guarantee single-valuedness for the phase of the wave functions and the path integral as it represents a map of the straight line on the circle, in which the harmonic functions appears

$$f(t_1) = \frac{1}{\cos \omega_0 t_1}; \qquad h(t_1) = 0; \qquad \dot{\lambda}(t_1) = \frac{\mathrm{i}\hbar\omega_0}{m}\tan \omega_0 t_1. \tag{54}$$

With this choice equation (6) for the case of a free particle yields

$$V_2(x_2,\, t_2) = V_1(x_1,\, t_1) - \frac{1}{2}m\omega_0^2 x_1^2 = 0 \tag{55}$$

and the phase $\Theta(x_1,\, t_1)$ in equation (5) yields

$$\Theta(x_1,\, t_1) = \frac{m\omega}{2\hbar}x_1^2 \tan \omega_0 t - \frac{\mathrm{i}}{2}\ln(\cos \omega_0 t_1). \tag{56}$$

Dropping again all sub-indices one finally obtains that the free Schrödinger equation

$$\left\{\mathrm{i}\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right\}\Psi(x,\, t) = 0$$

has a wave function; the solution of this free Schrödinger equation is of the form

$$\Psi(x,\,t) = \frac{(2^n n!)^{-\frac{1}{2}}}{\{\pi a_0^2(1+\omega_0^2 t^2)\}^{\frac{1}{4}}}\exp\left\{-\frac{x^2(1-\mathrm{i}\omega_0 t)}{2a_0^2(1+\omega_0^2 t^2)} - \mathrm{i}\left(n+\frac{1}{2}\right)\arctan(\omega_0 t)\right\}\mathbf{H}_n\left(\frac{x}{a_0(1+\omega_0^2 t^2)^{\frac{1}{2}}}\right)$$

where the fundamental length for this system now reads

$$a_0 = \left(\frac{\hbar}{m\omega_0}\right)^{\frac{1}{2}}.$$

In the case $\omega_0 = 0$, one obtains the usual wave functions of the harmonic oscillator. The meaning of ω_0 is not clear to us as it was the constant g in the previous case. Work in this direction is now in progress.

The last two expressions clearly show the equivalence between the free particle, and the damped and non-damped harmonic oscillator with time dependent frequency, even at the classical level. This equivalence is based upon the Lie group $SL(3,\,R)$ (see references [18] and [19]). The applications to the solutions of the harmonic oscillator either for the bound states and/or the Gaussian wave packets have been clearly shown in the recent references (see [20] and [21]).

6. Symmetries and path integrals

Let us go back to the beginning of the paper where the linear rigid transformations were defined

$$x_2 = f(t_1)x_1 + h(t_1)x_0 \tag{57a}$$

$$\mathrm{d}t_2 = f^2(t_1)\mathrm{d}t_1. \tag{57b}$$

In this section, we shall show that a different set of transformations exist such that $h(t_1) \neq 0$, defining a new way to eliminate the potentials of the previous section and at the same time yield the exact propagators of these two systems that, as it is well known, are exact in the stationary phase approximation [22]. The propagator arises also from the very same change of phase $\Theta(x_1,\,t_1)$ given by (5). This result unifies the symmetry method yielding both the solutions and the propagators as we shall show below for each of the potentials treated in section 4.

Case 1. The propagator for the linear potential

One needs a transformation of the form given by (23) such that transforms $V_1(x_1,\,t_1)$ into $V_2(x_2,\,t_2)$ using the expression (6) but keeping in mind that $V_1(x_1,\,t_1) = mgx_1$ and $V_2(x_2,\,t_2) = 0$. Under these conditions I shall prove that the transformation

$$f(t_1) = \frac{t_0}{(t_1 - t_0)} \tag{58a}$$

$$h(t_1) = -\frac{t_0}{(t_1 - t_0)} + \frac{gt_0}{2x_0}(t_1 - t_0) \tag{58b}$$

fulfils these conditions. From (24) it is trivial to see that

$$\frac{\dot{f}(t_1)}{f(t_1)} = -(t_1 - t_0)^{-1} \tag{59a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t_1}\left\{\frac{\dot{f}(t_1)}{f(t_1)}\right\} - \left(\frac{\dot{f}(t_1)}{f(t_1)}\right)^2 = 0 \tag{59b}$$

$$\frac{\dot{h}(t_1)}{f(t_1)} = (t_1 - t_0)^{-1} + \frac{g}{2x_0}(t_1 - t_0) \tag{59c}$$

$$\frac{\ddot{h}(t_1)}{f(t_1)} = -2(t_1 - t_0)^{-2} \tag{59d}$$

$$\frac{\ddot{h}(t_1)}{f(t_1)} - 2\left\{\frac{\dot{f}(t_1)}{f(t_1)}\right\}\left\{\frac{\dot{h}(t_1)}{f(t_1)}\right\} = \frac{g}{x_0}. \tag{59e}$$

Now we turn to (6) with the conditions $V_1(x_1, t_1) = mgx_1$ and $V_2(x_2, t_2) = 0$. Then we have

$$\begin{aligned} mgx_1 - \frac{m}{2}\left\{\frac{\mathrm{d}}{\mathrm{d}t_1}\left\{\frac{\dot{f}(t_1)}{f(t_1)}\right\} - \left(\frac{\dot{f}(t_1)}{f(t_1)}\right)^2\right\}x_1^2 - mx_0x_1\left\{\frac{\ddot{h}(t_1)}{f(t_1)} - 2\left\{\frac{\dot{f}(t_1)}{f(t_1)}\right\}\left\{\frac{\dot{h}(t_1)}{f(t_1)}\right\}\right\} \\ + \frac{mx_0^2}{2}\left\{\frac{\dot{h}(t_1)}{f(t_1)}\right\}^2 + i\frac{\hbar}{2}\frac{\dot{f}(t_1)}{f(t_1)} - m\dot{\lambda}(t_1) = 0. \end{aligned} \tag{60}$$

Inserting (25) in (26) and solving for $\dot{\lambda}(t_1)$ we obtain

$$\dot{\lambda}(t_1) = \frac{x_0^2}{2(t_1 - t_0)^2} + \frac{gx_0}{2} + \frac{g^2}{8}(t_1 - t_0)^2 - \frac{\mathrm{i}\hbar}{2m}\frac{1}{(t_1 - t_0)}. \tag{61}$$

Simple integration of (27) yields an expression for $\lambda(t_1)$, up to an arbitrary constant which plays no physical role and we can set to zero for convenience

$$\lambda(t_1) = -\frac{x_0^2}{2(t_1 - t_0)} + \frac{gx_0}{2}(t_1 - t_0) + \frac{g^2}{24}(t_1 - t_0)^3 - \frac{\mathrm{i}\hbar}{m}\ln(t_1 - t_0)^{\frac{1}{2}}. \tag{62}$$

Let us recall here the rule (5) of the phase transformation

$$\Theta(x_1, t_1) = \frac{m}{\hbar}\left\{\frac{1}{2}\frac{\dot{f}(t_1)}{f(t_1)}x_1^2 + \frac{\dot{h}(t_1)}{f(t_1)}x_1x_0 + \lambda(t_1)\right\} \tag{63}$$

substituting $f(t_1)$, $h(t_1)$ and $\lambda(t_1)$ from (24) and (25), we find

$$\Theta(x_1, t_1) = \frac{m}{\hbar}\left\{-\frac{x_1^2}{2(t_1 - t_0)} + \frac{x_1x_0}{(t_1 - t_0)} + \frac{gx_1}{2}(t_1 - t_0) - \frac{x_0^2}{2(t_1 - t_0)}\right. \tag{64}$$

$$\left. + \frac{gx_0}{2}(t_1 - t_0) + \frac{g^2}{24}(t_1 - t_0)^3 - \frac{\mathrm{i}\hbar}{m}\ln(t_1 - t_0)^{\frac{1}{2}}\right\} \tag{65}$$

and after some trivial algebra we finally obtain

$$\begin{aligned} \Theta(x_1, t_1) = \frac{1}{\hbar}\left\{-\frac{m(x_1 - x_0)^2}{2(t_1 - t_0)} + \frac{mg}{2}(x_1 + x_0)(t_1 - t_0)\right. \\ \left. + \frac{mg^2}{24}(t_1 - t_0)^3 - \mathrm{i}\hbar\ln(t_1 - t_0)^{\frac{1}{2}}\right\}. \end{aligned}$$

As was discussed at the beginning of the paper we learn that there is a new ‘wave function’ under the symmetry (25) which transforms as

$$\Psi_2(x_2, t_2) = \exp\{i\Theta(x_1, t_1)\}\Psi_1(x_1, t_1)$$

which converts the Schrödinger equation with a linear potential in the free particle but this time with a new phase, different from that of the previous section

$$\Psi_2(x_2, t_2) = \exp\left\{-\frac{i}{\hbar}\mathbf{S}_{cl}^{\mathbf{LP}}[x_1, t_1]\right\}\Psi_1(x_1, t_1)$$

which equals the inverse of the Feynmann propagator up to the normalizing factor $\left\{\frac{m}{2\pi i\hbar}\right\}^{\frac{1}{2}}$. In other words, a transformation of linear rigid transformations exists such that for the space and time coordinates it has the same effect as the action of the inverse Feynman propagator on the initial wave function of the system. I conclude that the action of the symmetry correctly describes not only the stationary wave function but also the time evolution of the system.

Case 2. The propagator for the harmonic oscillator

I proceed as in the previous case. However, now $V_1(x_1, t_1) = \frac{1}{2}m\omega_0^2 x_1^2$ and $V_2(x_2, t_2) = 0$. The transformation takes the form

$$f(t_1) = \frac{1}{\sin\omega_0 t_1} \tag{66a}$$

$$h(t_1) = -\cot\omega_0 t_1. \tag{66b}$$

From these functions we easily obtain

$$\frac{\dot{f}(t_1)}{f(t_1)} = -\omega_0\cot\omega_0 t_1 \tag{67a}$$

$$\frac{d}{dt_1}\left\{\frac{\dot{f}(t_1)}{f(t_1)}\right\} - \left(\frac{\dot{f}(t_1)}{f(t_1)}\right)^2 = \omega_0^2 \tag{67b}$$

$$\frac{\dot{h}(t_1)}{f(t_1)} = \frac{\omega_0}{\sin\omega_0 t_1} \tag{67c}$$

$$\frac{\ddot{h}(t_1)}{\dot{h}(t_1)} - 2\frac{\dot{f}(t_1)}{f(t_1)} = 0. \tag{67d}$$

Inserting the potentials $V_1(x_1, t_1) = \frac{1}{2}m\omega_0^2 x_1^2$ and $V_2(x_2, t_2) = 0$ in (6) we now have

$$\left\{\omega_0^2 - \frac{d}{dt_1}\left\{\frac{\dot{f}(t_1)}{f(t_1)}\right\} + \left(\frac{\dot{f}(t_1)}{f(t_1)}\right)^2\right\}x_1^2 - 2\left(\frac{\dot{h}(t_1)}{f(t_1)}\right)\left\{\frac{\ddot{h}(t_1)}{\dot{h}(t_1)} - 2\frac{\dot{f}(t_1)}{f(t_1)}\right\}x_0 x_1$$
$$+\left(\frac{\dot{h}(t_1)}{f(t_1)}\right)^2 x_0^2 + \frac{i\hbar}{m}\left(\frac{\dot{f}(t_1)}{f(t_1)}\right) - 2\lambda(t_1) = 0 \tag{68}$$

and again we find the expression for $\dot{\lambda}(t_1)$

$$\dot{\lambda}(t_1) = \frac{\omega_0^2 x_0^2}{2\sin^2\omega_0 t_1} - \frac{i\hbar\omega_0}{2m}\cot\omega_0 t_1. \tag{69}$$

Up to an inessential constant one obtains the expression for $\lambda(t_1)$

$$\lambda(t_1) = -\frac{\omega_0 x_0^2}{2}\cot\omega_0 t_1 - \frac{i\hbar}{2m}\ln\{\sin\omega_0 t_1\}. \tag{70}$$

The phase transformation is, as we already know

$$\Theta(x_1, t_1) = \frac{m}{\hbar}\left\{\frac{1}{2}\frac{\dot{f}(t_1)}{f(t_1)}x_1^2 + \frac{\dot{h}(t_1)}{f(t_1)}x_1x_0 + \lambda(t_1)\right\} \tag{71}$$

substituting $f(t_1)$, $h(t_1)$ y $\lambda(t_1)$ from (32) and (33), we find

$$\Theta(x_1, t_1) = -\frac{1}{\hbar}\left\{\frac{m\omega_0}{2\sin\omega_0 t_1}\{(x_1^2 + x_0^2)\cos\omega_0 t_1 - 2x_1x_0\} - i\hbar\ \ln\{\sin\omega_0 t_1\}^{-\frac{1}{2}}\right\} \tag{72}$$

and as in case 1

$$\Psi_2(x_2, t_2) = \exp\left\{-\frac{\mathrm{i}}{\hbar}\mathbf{S}_{cl}^{\mathbf{HO}}[x_1, t_1]\right\}\Psi_1(x_1, t_1).$$

As in the previous case one can see that a transformation of linear rigid transformations exists such that for the space and time coordinates it has the same effect as the action of the inverse Feynman propagator up to the normalizing factor $\left\{\frac{m\omega_0}{2\pi i\hbar}\right\}^{\frac{1}{2}}$ on the initial wave function of the system. I finally conclude that for systems which are exact in the stationary phase approximation, the action of the symmetry correctly describes not only the stationary wave function but also the time evolution of the system.

Acknowledgments

This research has been carried out with the partial support of MINECO under the project MAT2013-46308-C2-1-R and JCyL Project Number SA226U13.

Appendix

We collect in this appendix some of the main calculations leading to the transformed phase (5) and the fundamental relationship among the potentials (6). First, some simple changes in the partial derivatives under the linear rigid transformations are

$$\frac{\partial}{\partial t_1} = (\dot{f}x_1 + \dot{h}x_0)\frac{\partial}{\partial x_2} + f^2\frac{\partial}{\partial t_2}; \qquad \frac{\partial}{\partial x_1} = f\frac{\partial}{\partial x_2}; \qquad \frac{\partial^2}{\partial x_1^2} = f^2\frac{\partial^2}{\partial x_2^2}.$$

The Schrödinger equation in the variables of $\mathcal{M}_1\{x_1, t_1\}$ reads

$$\left\{\mathrm{i}\hbar\frac{\partial}{\partial t_1} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - V_1(x_1, t_1)\right\}\exp\{-\mathrm{i}\Theta(x_1, t_1)\}\Psi_2(x_2, t_2) = 0.$$

Inserting the changes in the partial derivatives written just above

$$\frac{\partial}{\partial x_1}\{\exp\{-\mathrm{i}\Theta(x_1, t_1)\}\Psi_2(x_2, t_2)\} = f\{-\mathrm{i}\frac{\partial\Theta}{\partial x_2}\Psi_2 + \frac{\partial\Psi_2}{\partial x_2}\}\exp\{-\mathrm{i}\Theta\}$$

$$\frac{\partial^2}{\partial x_1^2}\{\exp\{-\mathrm{i}\Theta(x_1, t_1)\}\Psi_2(x_2, t_2)\} = f^2\left\{-\left\{\left(\frac{\partial\Theta}{\partial x_2}\right)^2 + \mathrm{i}\frac{\partial^2\Theta}{\partial x_2^2}\right\}\Psi_2\right.$$

$$\left. - 2\mathrm{i}\frac{\partial\Theta}{\partial x_2}\frac{\partial\Psi_2}{\partial x_2} + \frac{\partial^2\Psi_2}{\partial x_2^2}\right\}\exp\{-\mathrm{i}\Theta\}$$

$$\frac{\partial}{\partial t_1}\{\exp\{-\mathrm{i}\Theta(x_1, t_1)\}\Psi_2(x_2, t_2)\} = \left\{(\dot{f}x_1 + \dot{h}x_0)\left\{-\mathrm{i}\frac{\partial\Theta}{\partial x_2}\Psi_2 + \frac{\partial\Psi_2}{\partial x_2}\right\}\right.$$
$$\left. + f^2\left\{-\mathrm{i}\frac{\partial\Theta}{\partial t_2}\Psi_2 + \frac{\partial\Psi_2}{\partial t_2}\right\}\right\}\exp\{-\mathrm{i}\Theta\}.$$

The Schrödinger equation is devoid of terms in the first derivatives of the space variables $\frac{\partial\Psi_2}{\partial x_2}$. Then the following condition must hold

$$f\frac{\partial\Theta}{\partial x_2} = \frac{m}{\hbar}\left\{\frac{\dot{f}}{f}x_1 + \frac{\dot{h}}{f}x_0\right\}$$

but

$$f\frac{\partial\Theta}{\partial x_2} = \frac{\partial\Theta}{\partial x_1}$$

thus, we have expression (5) in the text

$$\Theta(x_1, t_1) = \frac{m}{\hbar}\left\{\frac{\dot{f}(t_1)}{2f(t_1)}x_1^2 + \frac{\dot{h}(t_1)}{f(t_1)}x_1x_0 + \lambda(t_1)\right\}.$$

After $\Theta(x_1, t_1)$ is obtained, we go back to the Schrödinger equation and substitute the following expressions

$$\frac{\partial\Theta}{\partial x_1} = f\frac{\partial\Theta}{\partial x_2} = \frac{m}{\hbar}\left\{\frac{\dot{f}}{f}x_1 + \frac{\dot{h}}{f}x_0\right\}$$
$$\frac{\partial^2\Theta}{\partial x_1^2} = \frac{m}{\hbar}\left\{\frac{\dot{f}}{f}\right\}.$$

The result of this operation is

$$\mathrm{i}\hbar\frac{\partial\Psi_2}{\partial t_2} + \frac{\hbar^2}{2m}\frac{\partial^2\Psi_2}{\partial x_2^2} = \frac{1}{f^2}\left\{V_1(x_1, t_1) - \frac{m}{2f^2}\{\dot{f}x_1 + \dot{h}x_0\}^2 + \frac{.\hbar}{2}\frac{\dot{f}}{f} - \hbar\frac{\partial\Theta}{\partial t_2}\right\}\Psi_2.$$

Now we realize that the second member of the last equation must be equal to

$$V_2(x_2, t_2)\Psi_2.$$

With the help of the change of variables from the beginning of the appendix we easily go to

$$\frac{\partial\Theta}{\partial t_2} = \frac{1}{f^2}\left\{\frac{\partial\Theta}{\partial t_1} - \frac{1}{f}(\dot{f}x_1 + \dot{h}x_0)\frac{\partial\Theta}{\partial x_1}\right\} = -\frac{m}{\hbar f^4}(\dot{f}x_1 + \dot{h}x_0)^2 + \frac{1}{f^2}\frac{\partial\Theta}{\partial t_1}$$

and using the space and time derivatives for the phase $\Theta(x_1, t_1)$ t_1 already obtained, we finally obtain the transformation law for the potentials

$$V_2(x_2, t_2) = \frac{1}{f^2(t_1)}\left[V_1(x_1, t_1) - \frac{m}{2f^2(t_1)}\{f(t_1)\ddot{f}(t_1) - 2\dot{f}^2(t_1)\}x_1^2\right.$$
$$\left. - \frac{mx_0}{f^2(t_1)}\{f(t_1)\ddot{h}(t_1) - 2\dot{f}(t_1)\dot{h}(t_1)\}x_1 + \frac{mx_0^2}{2f^2(t_1)}\dot{h}^2(t_1) + \mathrm{i}\hbar\frac{\dot{f}(t_1)}{2f(t_1)} - m\dot{\lambda}(t_1)\right].$$

This is the expression (6) of the text.

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