Nonlinear Dyn (2017) 87:13–23
DOI 10.1007/s11071-016-2997-5

ORIGINAL PAPER

Symmetry computation and reduction of a wave model in 2 + 1 dimensions

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Received: 6 January 2016 / Accepted: 30 July 2016 / Published online: 12 August 2016

Abstract We present the iterative classical point symmetry analysis of a shallow water wave equation in $2+1$ dimensions and that of its corresponding non-isospectral, two-component Lax pair. A few reductions arise and are identified with celebrate equations in the Physics and Mathematics literature of nonlinear waves. We pay particular attention to the isospectral or non-isospectral nature of the reduced spectral problems.

Keywords Lie symmetries · Reduction · Nonlinear · Soliton · KdV equation · Painlevé test

1 Introduction

Invariance of a differential equation under a group of transformations is synonymous of existence of symmetry and, consequently, of conserved quantities [25]. Such invariance helps us to achieve partial or complete integration of the equation.

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A conserved quantity for a first-order differential equation can lead to its integration by quadrature, whilst for higher-order ones, it leads to a reduction of their order [28,31]. Many of the existing solutions to physical phenomena described by differential equations have been obtained through symmetry arguments [28,29,31]. Nevertheless, finding conserved quantities is a nontrivial task.

The most famous and established method for finding point symmetries is the classical Lie symmetry method (CLS) developed by Lie in 1881 [20–22,28,31]. Although the CLS represents a very powerful tool, it yields cumbersome calculations to be solved by hand. Notwithstanding, the increased number of available software packages for symbolic calculus has made of generalizations of the CLS analysis very tractable approaches to find conservation laws, explicit solutions, etc.

If we impose that symmetries leave certain submanifold invariant, we find the class of *conditional symmetries* or the so-called *nonclassical symmetry method* (NSM) introduced by Bluman and Cole [4] and later applied by many authors [3,9,15,26,27]. A remarkable difference between the CLS and NSM is that the latter provides us with no longer linear systems of differential equations from which to obtain the symmetries [5,36,37].

In this paper, we aim to perform the CLS on the so-called Bogoyavlenskii–Kadomtsev–Petviashvili equation [denoted as $(2+1)$-BKP equation, henceforth] which takes the form [13,14]

$$\left(u_{xt}+u_{xxxy}+8u_xu_{xy}+4u_{xx}u_y\right)_x+\sigma u_{yyy}=0,\\ \sigma=\pm 1. \tag{1}$$

This equation is a model for evolutionary shallow water waves represented by the scalar field $u(x, y, t)$ as the height of the wave. In [38], it was proved that this equation is a reduction of the well-known (3 + 1)-Kadomtsev–Petviashvili equation for the description of wave motion [12,18]. Notice that the (2 + 1)-BKP equation is also a modified version of the Calogero–Bogoyanlevski–Schiff equation [6,7,30].

Equation (1) is integrable according to the Painleve property (PP) and admits a Lax pair [14]. Results on solutions of solitonic nature for (1) were pursued through the *singular manifold method* [14,16,34,35]. In particular, *lump solitons*, or solutions which decay polynomially in all directions, were found. The interest of lump solutions roots in their nontrivial dynamics and interactions [13,17,24].

In particular, we focus on the study of its corresponding Lax pair in 2 + 1 dimensions [13]. It is a complex, two-component linear problem [14]

$$\psi_{xx}=-i\psi_y-2u_x\psi,\\ \psi_t=2i\psi_{yy}-4u_y\psi_x+(2u_{xy}+2i\omega_y)\psi \tag{2}$$

and its complex conjugate

$$\chi_{xx}=i\chi_y-2u_x\chi,\\ \chi_t=-2i\chi_{yy}-4u_y\chi_x+(2u_{xy}-2i\omega_y)\chi \tag{3}$$

with $\psi(x, y, t)$ and $\chi(x, y, t)$ being the eigenfunctions. The compatibility condition of the cross-derivatives ($\psi_{xxt}=\psi_{txx}$) in (2) retrieves (1) in the form.

$$u_{xt}+u_{xxxy}+8u_xu_{xy}+4u_{xx}u_y=\omega_{yy},\\ u_{yy}=\omega_{xy} \tag{4}$$

where we have introduced the auxiliary scalar field $\omega(x, y, t)$. We restrict ourselves to the case $\sigma=-1$. Results for $\sigma=1$ can be obtained by considering that if $u(x, y, t)$ is a solution for $\sigma=-1$, then $u(x, iy, it)$ is a solution for $\sigma=1$. The compatibility condition for ($\chi_{xxt}=\chi_{txx}$) in (3) yields (4) too.

Notice that the spectral parameter is not present in the Lax pair. This does not necessarily imply that it is an isospectral Lax pair. Indeed, there exists a gauge transformation which allows us to express the nonisospectral spectral linear problem as a spectral parameter-free linear problem. The converse is possible by introducing "λ" or *spectral parameter*, conveniently.

The importance of reduced spectral problems and reduced spectral parameters resides in the fact that in 1 + 1 dimensions, it is not usual to find nonisospectral versions. The nonisospectrality of 1 + 1-dimensional Lax pairs gives rise to some inconveniency, since the *inverse scattering transform* (IST) [1,2] can no longer be worked upon them, for example. If the IST cannot be used, the possibilities of solving the nonlinear equation through an associated spectral problem are diminished. On the other hand, many of the Lax pairs found in 2 + 1 dimensions are nonisospectral [15,16]. For this matter, we pay closer attention at the reductions of the associated spectral problem. In this way, we stress out the importance of finding isospectral Lax pairs in 1 + 1 dimensions and the surprising nature of nonisospectral ones.

In Sect. 2, we introduce the CLS for partial differential equations (PDEs) and apply it to the spectral problem (2) and its corresponding compatibility condition (4). In Sect. 3, we will obtain a classification of its possible reductions to 1 + 1 dimensions of the equation and the Lax pair, depending on different values of the arbitrary functions appearing in the obtained symmetries. We will identify six interesting reductions. Two of them will be nontrivial. One of such nontrivial reduction corresponds with the celebrated Korteweg de Vries equation (KdV equation) [10,19,23]. The second reduction found will be submitted to a second CLS calculation. So, Sect. 4 shall be devoted to the classical symmetry computation of the aforementioned second nontrivial reduction in 1 + 1 dimensions. A list of four reductions to ODEs will be displayed, considering different values for the constants of integration appearing in the symmetry computation. To conclude, we shall enclose a summary of the most relevant results found throughout the "iterative symmetry search and reduction" procedure.

2 The classical symmetry approach

A priori, we propose the most general Lie point symmetry transformation in which the coefficients of the infinitesimal generator can depend on any dependent or independent variable

$$
\begin{aligned}
x &\to x + \epsilon \xi_1(x, y, t, u, \omega) + O(\epsilon^2),\\
y &\to y + \epsilon \xi_2(x, y, t, u, \omega) + O(\epsilon^2),\\
t &\to t + \epsilon \xi_3(x, y, t, u, \omega) + O(\epsilon^2),\\
u &\to u + \epsilon \eta_u(x, y, t, u, \omega) + O(\epsilon^2),\\
\omega &\to \omega + \epsilon \eta_\omega(x, y, t, u, \omega) + O(\epsilon^2).
\end{aligned} \tag{5}
$$

If we search for general Lie point transformation of the Lax pair too, the spectral functions must be transformed accordingly with

$$
\begin{aligned}
\psi &\to \psi + \epsilon \eta_\psi(x, y, t, u, \omega, \psi, \chi) + O(\epsilon^2),\\
\chi &\to \chi + \epsilon \eta_\chi(x, y, t, u, \omega, \psi, \chi) + O(\epsilon^2).
\end{aligned} \tag{6}
$$

Associated with this transformation, there exists an infinitesimal generator

$$
X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial t} + \eta_u \frac{\partial}{\partial u} + \eta_\omega \frac{\partial}{\partial \omega} + \eta_\psi \frac{\partial}{\partial \psi} + \eta_\chi \frac{\partial}{\partial \chi}, \tag{7}
$$

where the subscripts in η_p have been added according to the field "p" to which each η is associated. This transformation must leave (2), (3) and (4), invariant. From now, since Eqs. (2) and (3) are equivalent, we shall exclude (3) from our calculus as a matter of simplification. Our results obtained for ψ can be similarly extrapolated for χ.

To proceed with CLS, we follow the next steps

1. Introduce the infinitesimal transformation (5) and its further derivatives in Eqs. (2) and (4).
2. Select the linear terms in ϵ and set them equal to zero. The zero order in ϵ retrieves the original equations.
3. Substitute the values of $\psi_{xx}, \psi_t, u_{xt}, u_y$ from (2) and (4), correspondingly. Higher-order derivatives in these terms must be introduced according to these expressions.
4. From the steps above, we obtain an overdetermined system of differential equations for $\xi_1, \xi_2, \xi_3, \eta_u, \eta_\omega, \eta_\psi$ by setting equal to zero terms in different orders of the derivatives of the fields, if we pursue Lie point symmetries.

Steps 1–5 are nontrivial, leading to cumbersome equations which have been manipulated with a symbolic calculus package. In our case, we make use of MAPLE©.

We extend transformation (5) to first-, second- and higher-order derivatives appearing in Eq. (4),

$$
\begin{aligned}
u_x &\to u_x + \epsilon \eta_u^{[x]} + O(\epsilon^2),\\
u_y &\to u_y + \epsilon \eta_u^{[y]} + O(\epsilon^2),\\
\omega_x &\to \omega_x + \epsilon \eta_\omega^{[x]} + O(\epsilon^2),\\
\omega_y &\to \omega_y + \epsilon \eta_\omega^{[y]} + O(\epsilon^2),\\
u_{xx} &\to u_{xx} + \epsilon \eta_u^{[xx]} + O(\epsilon^2),\\
u_{xy} &\to u_{xy} + \epsilon \eta_u^{[xy]} + O(\epsilon^2),\\
u_{xt} &\to u_{xt} + \epsilon \eta_u^{[xt]} + O(\epsilon^2),\\
\omega_{yy} &\to \omega_{yy} + \epsilon \eta_\omega^{[yy]} + O(\epsilon^2),\\
u_{xxxy} &\to u_{xxxy} + \epsilon \eta_u^{[xxxy]} + O(\epsilon^2).
\end{aligned} \tag{8}
$$

And derivatives appearing in the Lax pair (2)

$$
\begin{aligned}
\psi_x &\to \psi_x + \epsilon \eta_\psi^{[x]} + O(\epsilon^2),\\
\psi_y &\to \psi_y + \epsilon \eta_\psi^{[y]} + O(\epsilon^2),\\
\psi_t &\to \psi_t + \epsilon \eta_\psi^{[t]} + O(\epsilon^2),\\
\psi_{xx} &\to \psi_{xx} + \epsilon \eta_\psi^{[xx]} + O(\epsilon^2),\\
\psi_{yy} &\to \psi_{yy} + \epsilon \eta_\psi^{[yy]} + O(\epsilon^2).
\end{aligned} \tag{9}
$$

The prolongations needed for Eq. (4) are $\eta_u^{[xxxy]}$, $\eta_u^{[xt]}$, $\eta_u^{[xy]}$, $\eta_u^{[xx]}$, $\eta_\omega^{[yy]}$, $\eta_u^{[x]}$, $\eta_u^{[y]}$, $\eta_\omega^{[x]}$, $\eta_\omega^{[y]}$ and for the Lax pair (2), the prolongations are $\eta_\psi^{[yy]}$, $\eta_\psi^{[xx]}$, $\eta_\psi^{[t]}$, $\eta_\psi^{[y]}$, $\eta_\psi^{[x]}$. These prolongations can be calculated according to the Lie method explained in textbooks [31]. From the original equations, we make use of

$$
\begin{aligned}
\omega_{yy} &= u_{xt} + u_{xxxy} + 8u_x u_{xy} + 4u_{xx} u_y,\\
u_{yy} &= \omega_{xy},
\end{aligned} \tag{10}
$$

and the original the Lax pair

$$
\begin{aligned}
\psi_{xx} &= -i\psi_y - 2u_x \psi,\\
\psi_t &= 2i\psi_{yy} - 4u_y \psi_x + (2u_{xy} + 2i\omega_y)\psi.
\end{aligned} \tag{11}
$$

Also, their further derivatives must be computed using the given expressions.

Introducing such relations we arrive at the classical Lie symmetries

$$
\begin{aligned}
\xi_1 &= \frac{\dot{A}_3(t)}{4} x + A_1(t),\\
\xi_2 &= \frac{\dot{A}_3(t)}{2} y + A_2(t),\\
\xi_3 &= A_3(t),
\end{aligned}
$$

Table 1 Reductions for 2 + 1-BKP

Case I: $A_3(t) \neq 0$			Case II: $A_3(t) = 0$		
1.	$A_1(t) \neq 0$	$A_2(t) \neq 0$	1.	$A_1(t) \neq 0$	$A_2(t) \neq 0$
2.	$A_1(t) \neq 0$	$A_2(t) = 0$	2.	$A_1(t) \neq 0$	$A_2(t) = 0$
3.	$A_1(t) = 0$	$A_2(t) \neq 0$	3.	$A_1(t) = 0$	$A_2(t) \neq 0$

$$\begin{aligned}\eta_u &= -\frac{\dot{A}_3(t)}{4}u + \frac{\dot{A}_2(t)}{8}x + \frac{\ddot{A}_3(t)}{16}xy \\ &\quad + \frac{\dot{A}_1(t)y}{4} + B_1(t), \\ \eta_\omega &= -\frac{\dot{A}_3(t)}{2}w + \frac{\dot{A}_1(t)}{4}x + \frac{\ddot{A}_3(t)}{32}x^2 \\ &\quad + B_3(t)y + \frac{\ddot{A}_2(t)}{16}y^2+, \\ &\quad + \frac{\dddot{A}_3(t)}{96}y^3 + B_2(t), \\ \eta_\psi &= \left[-2\lambda - \frac{\dot{A}_3(t)}{8} + i\left(\frac{\dot{A}_2(t)}{4}y \right.\right. \\ &\quad \left.\left. + \frac{\ddot{A}_3(t)}{16}y^2 + 2\int B_3(t)\mathrm{d}t\right)\right]\psi. \end{aligned} \tag{12}$$

These symmetries depend on a constant λ and six arbitrary functions of time, $A_1(t)$, $A_2(t)$, $A_3(t)$ and $B_1(t)$, $B_2(t)$, $B_3(t)$, which shall serve us as a way to classify the possible reductions. Indeed, we have the possible reductions attending to (Table 1).

3 Reduction to 1 + 1 dimensions

To reduce the problem, we have to solve the Lagrange-Charpit system by the method of characteristics, that is integration of the characteristic system [28,31]

$$\frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{\mathrm{d}t}{\xi_3} = \frac{du}{\eta_u} = \frac{d\omega}{\eta_\omega} = \frac{d\psi}{\eta_\psi}. \tag{13}$$

We shall use the next notation for the reduced variables

$$x, y, t \to x_1, x_2 \tag{14}$$

and for the reduced fields

$$\begin{aligned}\omega(x, y, t) &\to \Omega(x_1, x_2), \\ u(x, y, t) &\to U(x_1, x_2), \\ \psi(x, y, t) &\to \Phi(x_1, x_2).\end{aligned} \tag{15}$$

As a matter of simplification, we shall drop the dependency on (x_1, x_2) of certain fields in the forthcoming expressions.

- *Case I.1.* $A_3(t) \neq 0, A_1 \neq 0, A_2 \neq 0$
  - Reduced variables

$$x_1 = \frac{x}{A_3(t)^{1/4}} - \int \frac{A_1(t)}{A_3(t)^{5/4}}\mathrm{d}t, \quad x_2 = \frac{y}{A_3(t)^{1/2}} - \int \frac{A_2(t)}{A_3(t)^{3/2}}\mathrm{d}t.$$

  - Reduced fields

$$\begin{aligned}u(x, y, t) &= \frac{U(x_1, x_2) + \int F_1(t)\mathrm{d}t}{A_3(t)^{1/4}} \\ &\quad + \frac{\dot{A}_3(t)xy + 2A_2(t)x + 4A_1(t)y}{16A_3(t)}, \\ \omega(x, y, t) &= \frac{\Omega(x_1, x_2) + \int F_2(t)\mathrm{d}t}{A_3(t)^{1/2}} \\ &\quad + \frac{2A_3(t)\ddot{A}_3(t) - \dot{A}_3(t)^2}{192A_3(t)^2}y^3 \\ &\quad + \frac{2A_3(t)\dot{A}_2(t) - A_2(t)\dot{A}_3(t)}{32A_3(t)^2}y^2 \\ &\quad + \left(\frac{\int B_3(t)\mathrm{d}t}{A_3(t)} - \frac{A_2(t)^2}{16A_3(t)^2}\right)y \\ &\quad + \frac{\dot{A}_3(t)}{32A_3(t)}x^2 + \frac{A_1(t)}{4A_3(t)}x,\end{aligned}$$

  where we have used the definitions

$$\begin{aligned}F_1(t) &= \frac{B_1(t)}{A_3(t)^{3/4}} - \frac{3A_1(t)A_2(t)}{8A_3(t)^{7/4}}, \\ F_2(t) &= \frac{B_2(t)}{A_3(t)^{1/2}} - \frac{A_2(t)\int B_3(t)\mathrm{d}t}{A_3(t)^{3/2}} \\ &\quad + \frac{A_2(t)^3}{16A_3(t)^{5/2}} - \frac{A_1(t)^2}{4A_3(t)^{3/2}}.\end{aligned}$$

  - Reduced equation

$$\begin{aligned}&U_{x_2x_2} - \Omega_{x_1x_2} = 0, \\ &\Omega_{x_2x_2} = U_{x_1x_1x_1x_2} + 8U_{x_1x_2}U_{x_1} + 4U_{x_1x_1}U_{x_2}.\end{aligned} \tag{16}$$

  These two equations can be summarized as

$$U_{x_2x_1x_1x_1x_1} + 4\Big((U_{x_1}U_{x_2})_{x_1} + U_{x_1}U_{x_1x_2}\Big)_{x_1} - U_{x_2x_2x_2} = 0, \tag{17}$$

which appears in [32,33] and has *multiple soliton solutions* [39].

- Reduced eigenfunction

$$\psi(x,y,t) = \frac{\Phi(x_1,x_2)}{A_3(t)^{1/8}} \times \exp\left[i\left(\frac{\dot{A}_3(t)y^2+4A_2(t)y}{16A_3(t)}\right) + i\int F_3(t)\mathrm{d}t\right].$$

where

$$F_3(t) = \frac{2\int B_3(t)\mathrm{d}t}{A_3(t)} + \frac{2i\lambda}{A_3(t)} - \frac{A_2(t)^2}{4A_3(t)^2}.$$

- Reduced Lax pair

$$\Phi_{x_1x_1} + i\Phi_{x_2} + 2U_{x_1}\Phi = 0,$$
$$i\Phi_{x_2x_2} - 2U_{x_2}\Phi_{x_1} + \left(U_{x_1x_2} + i\Omega_{x_2} + \lambda\right)\Phi = 0. \tag{18}$$

Notice that "λ" plays the role of the spectral parameter in $1+1$ dimensions.

- Cases I.2. and I.3. will be omitted for being equivalent to I.1.
- *Case II.1.* $A_3(t) = 0$, $A_1(t) \neq 0$, $A_2(t) \neq 0$
  - Reduced variables

$$x_1 = \frac{A_2(t)x - A_1(t)y}{A_2(t)^{3/2}} - \int F_1(t)\mathrm{d}t$$
$$x_2 = \int \frac{A_1(t)}{A_2(t)^{5/2}}\mathrm{d}t$$

  - Reduced fields

$$u(x,y,t) = \frac{U(x_1,x_2) + \left(\int B_3(t)\mathrm{d}t - \frac{A_1(t)^2}{8A_2(t)}\right)x_1}{\sqrt{A_2(t)}} + \left(\frac{B_1(t)}{A_2(t)} + \frac{\dot{A}_2(t)}{8A_2(t)}x\right)y + \left(2\frac{\dot{A}_1(t)}{A_2(t)} - \frac{A_1(t)\dot{A}_2(t)}{A_2(t)^2}\right)\frac{y^2}{16}$$

$$w(x,y,t) = -A_1(t)\frac{\Omega(x_1,x_2)}{A_2(t)^{3/2}} + \frac{\dot{A}_2(t)}{16}x_1^2 + \frac{\ddot{A}_2(t)}{48A_2(t)}y^3 + \left(4\frac{B_3(t)}{A_2(t)} - \frac{A_1(t)\dot{A}_1(t)}{A_2(t)^2}\right)\frac{y^2}{8} + \frac{B_2(t)}{A_2(t)}y + \frac{\dot{A}_1(t)}{4A_2(t)}xy$$

with

$$F_1(t) = \frac{5}{2}\frac{A_1(t)^3}{A_2(t)^{7/2}} + 4\frac{B_1(t)}{A_2(t)^{3/2}} - 12\frac{A_1(t)}{A_2(t)^{5/2}}\int B_3(t)\mathrm{d}t$$

  - Reduced equation

$$U_{x_1x_1} - \Omega_{x_1x_1} = 0 \Rightarrow \Omega_{x_1} = U_{x_1} + G(x_2)$$
$$U_{x_1x_1x_1x_1} + 12U_{x_1x_1}U_{x_1} - U_{x_1x_2} = 0.$$

where $G(x_2)$ is an arbitary function of x_2. The above system can be equivalently rewritten as

$$U_{x_1x_1x_1x_1x_1} + 12U_{x_1x_1}^2 + 12U_{x_1}U_{x_1x_1x_1} - U_{x_1x_1x_2} = 0. \tag{19}$$

This reduced equation corresponds with the potential KdV equation in $1+1$ dimensions [11]. Therefore, we can conclude that the $(2+1)$-BKP equation is a generalization of the potential KdV equation to $2+1$ dimensions.

  - Reduced Eigenfunction

$$\Psi(x,y,t) = \frac{\Phi(x_1,x_2)}{A_2(t)^{1/4}} \times \exp\left[i\left(\frac{A_1(t)}{2A_2(t)^{1/2}}x_1 + \int F_2(t)\mathrm{d}t\right)\right] \times \exp\left[i\left(\frac{\dot{A}_2(t)}{8A_2(t)}y^2 + 2\frac{\left(\int B_3(t)\mathrm{d}t + i\lambda\right)}{A_2(t)}y\right)\right]$$

where

$$F_2(t) = \frac{2B_2(t)}{A_2(t)} + \frac{A_1(t)^4}{2A_2(t)^4} + 2\frac{A_1(t)^2}{A_2(t)^3}\left(G(x_2(t)) + 4i\lambda\right) + \frac{\dot{A}_1(t)}{2A_2(t)^{1/2}}\int F_1(t)\mathrm{d}t - 8\left(\frac{\int B_3(t)\mathrm{d}t + i\lambda}{A_2(t)}\right)^2$$

  - Reduced Lax pair

$$\Phi_{x_1x_1} - 2(i\lambda - U_{x_1})\Phi = 0 \tag{20}$$
$$\Phi_{x_2} - 4(2i\lambda + U_{x_1})\Phi_{x_1} + 2U_{x_1x_1}\Phi = 0 \tag{21}$$

which is the Lax pair corresponding with the KdV equation.
Again, "λ" plays the role of the spectral parameter.

- *Case II.2.* $A_3(t)=0,\ A_1(t)\neq 0,\ A_2(t)=0$
  - Reduced variables

$$x_1 = y, \quad x_2 = t.$$

  - Reduced fields

$$\begin{aligned}
u(x,y,t) &= \frac{iA_1(t)U(x_1,x_2)}{8\left(\int B_3(t)\mathrm{d}t + i\lambda\right)} + \frac{B_3(t)}{2A_1(t)}x_1^2 \\
&\quad + \left(\frac{B_1(t)}{A_1(t)} + \frac{\dot{A}_1(t)}{4A_1(t)}y\right)x \\
w(x,y,t) &= i\frac{\Omega(x_1,x_2)}{2} + \frac{\dot{A}_1(t)}{8A_1(t)}x^2 \\
&\quad + \left(\frac{B_2(t)}{A_1(t)} + \frac{B_3(t)}{A_1(t)}y\right)x \\
&\quad + \frac{\dot{A}_1(t)^2 + A_1(t)\ddot{A}_1(t)}{24A_1(t)^2}x_1^3 \\
&\quad + \frac{B_1(t)\dot{A}_1(t) + A_1(t)\dot{B}_1(t)}{2A_1(t)^2}x_1^2
\end{aligned}$$

These reduced fields lead to a trivial reduction of the equations.

  - Reduced equation

$$U_{x_1x_1} = 0, \qquad \Omega_{x_1x_1} = 0$$

  - Reduced eigenfunction

$$\begin{aligned}
\Psi(x,y,t) &= \frac{\Phi(x_1,x_2)}{\sqrt{A_1(t)}} \\
&\quad \times \exp\left[i\left(\frac{2\left(\int B_3(t)\mathrm{d}t + i\lambda\right)}{A_1(t)}x - 8\int F_1(t)\mathrm{d}t\right)\right] \\
&\quad \times \exp\left[i\left(\frac{\dot{A}_1(t)}{4A_1(t)}x_1^2 \right.\right. \\
&\qquad \left.\left. + 2\frac{A_1(t)B_1(t) - 2\left(\int B_3(t)\mathrm{d}t + i\lambda\right)^2}{A_1(t)^2}x_1\right)\right] \\
F_1(t) &= \left[-\frac{B_1(t)}{A_1(t)} + 2\left(\frac{\int B_3(t)\mathrm{d}t + i\lambda}{A_1(t)}\right)^2\right]^2
\end{aligned}$$

  - Reduced Lax pair

$$\begin{aligned}
&\Phi_{x_1} = 0, \\
&\Phi_{x_2} - \left(U_{x_1} - \Omega_{x_1}\right)\Phi = 0
\end{aligned} \tag{22}$$

- *Case II.3.* $A_3(t)=0,\ A_1(t)=0,\ A_2(t)\neq 0$
  - Reduced variables

$$x_1 = \frac{x}{\sqrt{A_2(t)}} - 4\int\frac{B_1(t)}{A_2(t)^{3/2}}\mathrm{d}t \quad x_2 = t,$$

  - Reduced fields

$$\begin{aligned}
u(x,y,t) &= \frac{U(x_1,x_2) + x_1\int B_3(t)\mathrm{d}t}{\sqrt{A_2(t)}} \\
&\quad + \left(\frac{B_1(t)}{A_2(t)} + \frac{\dot{A}_2(t)}{8A_2(t)}x\right)y \\
w(x,y,t) &= \Omega(x_1,x_2) + \frac{A_2(t)}{16}x_1^2 + \frac{\ddot{A}_2(t)}{48A_2(t)}y^3 \\
&\quad + \frac{B_3(t)}{2A_2(t)}y^2 + \frac{B_2(t)}{A_2(t)}y
\end{aligned}$$

These reduced fields lead to trivial a reduction of the equation.

  - Reduced equation

$$U_{x_1x_2} = 0 \tag{23}$$

  - Reduced eigenfunction

$$\begin{aligned}
\Psi(x,y,t) &= \frac{\Phi(x_1,x_2)}{A_2(t)^{1/4}} \\
&\quad \exp\left[i\left(\frac{\dot{A}_2(t)}{8A_2(t)}y^2 + 2\frac{\int B_3(t)\mathrm{d}t + i\lambda}{A_2(t)}y \right.\right. \\
&\qquad \left.\left. + \int F_3(t)\mathrm{d}t\right)\right] \\
F_3(t) &= \frac{2B_2(t)}{A_2(t)} - 8\left(\frac{\int B_3(t)\mathrm{d}t + i\lambda}{A_2(t)}\right)^2
\end{aligned}$$

  - Reduced Lax pair

$$\begin{aligned}
&\Phi_{x_2} = 0 \\
&\Phi_{x_1x_1} + 2\left(U_{x_1} - i\lambda\right)\Phi = 0
\end{aligned}$$

3.1 Reduction of a Lax pair in 1 + 1 dimensions

Let us now study the nontrivial reduction I.1. obtained in the past section. We consider this reduction of interest from a possible physical viewpoint. In this way, we aim to perform another symmetry search on the equation and its corresponding Lax pair in 1+1 dimensions.

We reconsider the equation given in (17)

$$U_{x_2x_2} = \Omega_{x_1x_2},$$
$$U_{x_1x_1x_1x_2} + 8U_{x_1x_2}U_{x_1} + 4U_{x_1x_1}U_{x_2} - \Omega_{x_2x_2} = 0, \tag{24}$$

which is integrable in the Painlevé sense and possesses an associated linear spectral problem or Lax pair (18), which takes the form

$$\Phi_{x_1x_1} + i\Phi_{x_2} + 2U_{x_1}\Phi = 0,$$
$$i\Phi_{x_2x_2} - 2U_{x_2}\Phi_{x_1} + \left(U_{x_1x_2} + i\Omega_{x_2} + \lambda\right)\Phi = 0. \tag{25}$$

whose compatibility condition ($\Phi_{x_1x_1x_2x_2} = \Phi_{x_2x_2x_2x_1}$) recovers (24). This Lax pair presents a constant parameter "λ" that plays the role of the spectral parameter.

We aim at studying its classical Lie point symmetries and further reduction under the action of the symmetries.

We propose a general transformation in which the infinitesimal generator depends on any independent and dependent variables as

$$x_1 \to x_1 + \epsilon\xi_1(x_1, x_2, U, \Omega) + O(\epsilon^2),$$
$$x_2 \to x_2 + \epsilon\xi_2(x_1, x_2, U, \Omega) + O(\epsilon^2),$$
$$\lambda \to \lambda + \epsilon\eta_\lambda(x_1, x_2, \lambda, \Phi) + O(\epsilon^2),$$
$$U \to U + \epsilon\eta_U(x_1, x_2, U, \Omega) + O(\epsilon^2),$$
$$\Omega \to \Omega + \epsilon\eta_\Omega(x_1, x_2, U, \Omega) + O(\epsilon^2). \tag{26}$$

Here, we can see that we have considered λ as an independent variable in order to make the reductions properly. If we want to achieve symmetries of equation (24) and those of the corresponding Lax pair (25) at the same time, we must include the transformed eigenfunctions

$$\Phi \to \Phi + \epsilon\eta_\Phi(x_1, x_2, U, \Omega, \lambda, \Phi) + O(\epsilon^2),$$

where we have only specified the transformation for one of the eigenfunctions, since the complex conjugate version was not considered in the former reductions. Similar results can be obtained for the complex conjugate by extension of previous results.

By definition of symmetry, transformation (26) must leave invariant equation (24) and its associated Lax pair (25). The associated symmetry vector field has the expression

$$X = \xi_1\frac{\partial}{\partial x_1} + \xi_2\frac{\partial}{\partial x_2} + \eta_\lambda\frac{\partial}{\partial\lambda} + \eta_U\frac{\partial}{\partial U} + \eta_\Omega\frac{\partial}{\partial\Omega} + \eta_\Phi\frac{\partial}{\partial\Phi}. \tag{27}$$

From the terms in $\epsilon = 0$ we retrieve the original equations,

$$U_{x_2x_2} = \Omega_{x_1x_2},$$
$$U_{x_1x_1x_1x_2} = \Omega_{x_2x_2} - 8U_{x_1x_2}U_{x_1} - 4U_{x_1x_1}U_{x_2},$$
$$\Phi_{x_1x_1} = -i\Phi_{x_2} - 2U_{x_1}\Phi,$$
$$i\Phi_{x_2x_2} = 2U_{x_2}\Phi_{x_1} - \left(U_{x_1x_2} - i\Omega_{x_2} - \lambda\right)\Phi \tag{28}$$

that shall be used in the forthcoming steps.

First introduce transformation (26) into the system of differential equations in (24), (25) and set the linear term in ϵ equal to zero. Introduce the needed prolongations for Eq. (24), that are $\eta_U^{[x_1x_1x_1x_2]}$, $\eta_U^{[x_1x_2]}$, $\eta_U^{[x_1x_1]}$, $\eta_U^{[x_2x_2]}$, $\eta_U^{[x_1]}$, $\eta_U^{[x_2]}$, $\eta_\Omega^{[x_2x_2]}$, $\eta_\Omega^{[x_1x_2]}$, $\eta_\Omega^{[x_2]}$, and the prolongations neeeded for the Lax pair (25), that are $\eta_\Phi^{[x_1x_1]}$, $\eta_\Phi^{[x_2x_2]}$, $\eta_\Phi^{[x_1]}$, $\eta_\Phi^{[x_2]}$, calculated following Lie's formula [31] and $U_{x_2x_2}$, $U_{x_1x_1x_1x_2}$, $\Phi_{x_1x_1}$, $\Phi_{x_2x_2}$ from (28).

We come up with the classical Lie point symmetries

$$\xi_1(x_1, x_2, U, \Omega) = \frac{1}{2}k_1x_1 + k_2,$$
$$\xi_2(x_1, x_2, U, \Omega) = k_1x_2 + k_3,$$
$$\eta_\lambda = -2k_1\lambda + k_4,$$
$$\eta_u(x_1, x_2, U, \Omega) = -\frac{1}{2}k_1U + k_5,$$
$$\eta_\Omega(x_1, x_2, U, \Omega) = -k_1\Omega + ik_4x_2 + k_6(x_1),$$
$$\eta_\Phi(x_1, x_2, \lambda, U, \Omega, \Phi, \lambda) = B(\lambda)\Phi. \tag{29}$$

These symmetries depend on 5 arbitrary constants of integration k_1, k_2, k_3, k_4, k_5 and two arbitrary functions $k_6(x_1)$ and $B(\lambda)$ (Table 2).

We introduce the following notation for the reduced variables. In this case, λ is an independent variable.

$$x_1, x_2, \lambda \to z, \Lambda \tag{30}$$

Table 2 Reductions for 1 + 1-BKP equation

Case I: $k_1 \neq 0$			Case II: $k_1 = 0$		
1.	$k_2 \neq 0$	$k_3 \neq 0$	1.	$k_2 \neq 0$	$k_3 \neq 0$
			2.	$k_2 \neq 0$	$k_3 = 0$
			3.	$k_2 = 0$	$k_3 \neq 0$

and the reduced fields and eigenfunctions

$$
\begin{aligned}
&U(x_1, x_2) \to V(z),\\
&\Omega(x_1, x_2) \to W(z),\\
&\Phi(x_1, x_2) \to \varphi(z, \Lambda).
\end{aligned} \tag{31}
$$

We find the following reductions

- *Case I.1.* $k_1 \neq 0$
  - Reduced variables

$$
\begin{aligned}
z &= \frac{k_1(k_1x_2 + k_3)}{(k_1x_1 + 2k_2)^2},\\
\Lambda &= k_1^{-5}(2k_1\lambda - k_4)(k_1x_1 + 2k_2)^4.
\end{aligned}
$$

  - Reduced fields

$$
\begin{aligned}
U(x_1, x_2) &= k_1\frac{V(z)}{(k_1x_1 + 2k_2)},\\
\Omega(x_1, x_2) &= \frac{ik_4}{2k_1^3}(k_1x_1 + 2k_2)^2 z\\
&\quad - \frac{k_1^2}{2(k_1x_1 + 2k_2)^2}\frac{W(z)}{z}.
\end{aligned}
$$

  - Reduced Eigenfunction

$$
\Phi(x_1, x_2, \lambda) = \varphi(z, \Lambda)e^{\int \frac{B(\lambda)}{k_4 - 2k_1\lambda}d\lambda}
$$

  - Reduced Lax pair

$$
\begin{aligned}
&2z^2\varphi_{zz} + 8iz^2V_z(2\Lambda\varphi_\Lambda - z\varphi_z)\\
&\quad + \Big[W - z(W_z - 6izV_z)\\
&\qquad + iz^2(4zV_{zz} - \Lambda)\Big]\varphi = 0,\\
&16\Lambda^2\varphi_{\Lambda\Lambda} - 16z\Lambda\varphi_{z\Lambda}\\
&\quad + 4\Lambda(3-8iz^2V_z)\varphi_\Lambda + (i + 6z + 16iz^3V_z)\varphi_z\\
&\quad - 2\Big[W - zW_z + V + 2z(1 + 3iz)V_z\\
&\quad + iz^2(4zV_{zz} - \Lambda)\Big]\varphi = 0.
\end{aligned}
$$

In this case, a constant "Λ" appears and plays the role of a spectral parameter.

  - Reduced equations

$$
\begin{aligned}
&W_{zz} = V_{zz},\\
&16z^6V_{zzzz} + 144z^5V_{zzz}\\
&\quad + 8z^3(15 - 8V)V_z + 2zW_z - 2W\\
&\quad - (1 - 300z^2 + 32z^2V + 96z^3V_z)z^2V_{zz}\\
&\quad - 176z^4V_z^2 = 0.
\end{aligned}
$$

These equations can be integrated as

$$
\begin{aligned}
&W = a_1z + a_2\\
&16z^5V_{zzz} + 80z^4V_{zz} - 48z^4V_z^2\\
&\quad + z^3(-32V + 60)V_z\\
&\quad - zV_z + 2V + 2a_2 - a_3z = 0.
\end{aligned}
$$

where a_1, a_2, a_3 are constants.

- *Case II.1.* $k_1 = 0, k_2 \neq 0, k_3 \neq 0, k_4 \neq 0$
  - Reduced variables

$$
\begin{aligned}
z &= \frac{k_2}{k_3}\left(\frac{k_2}{k_3}x_2 - x_1\right)\\
\Lambda &= \frac{k_2}{k_3}\left(\frac{k_2}{k_4}\lambda - x_1\right).
\end{aligned}
$$

  - Reduced fields

$$
\begin{aligned}
U(x_1, x_2) &= \frac{k_2}{k_3}V(z) + \frac{k_5}{k_2}x_1,\\
\Omega(x_1, x_2) &= -\frac{k_2^2}{k_3^2}W(z) + i\frac{k_4k_3^2}{k_2^3}\left(z + \frac{k_2}{2k_3}x_1\right)x_1\\
&\quad + \frac{1}{k_2}\int k_6(x_1)dx_1.
\end{aligned}
$$

  - Reduced Eigenfunction

$$
\Phi(x_1, x_2, \lambda) = \varphi(z, \Lambda)e^{\frac{1}{k_4}\int B(\lambda)d\lambda}.
$$

  - Reduced Lax pair

$$
\begin{aligned}
&\varphi_{\Lambda\Lambda} + 2\varphi_{z\Lambda} + \varphi_{zz} + i\varphi_z + 2(C_1 - V_z)\varphi = 0,\\
&(-iC_2\Lambda - W_z + iV_{zz})\varphi\\
&\quad - 2i(\varphi_\Lambda + \varphi_z)V_z + \varphi_{zz} = 0.
\end{aligned} \tag{32}
$$

In this reduced spectral problem, the constant "Λ" plays the role of the spectral parameter.

- Reduced equation

$$W_{zz} = V_{zz} - iC_2,$$
$$V_{zzzz} + 4\left(2C_1 - 3V_z\right)V_{zz} - W_{zz} = 0. \quad (33)$$

where $C_1 = \frac{k_3^2 k_5}{k_2^3}$ and $C_2 = \frac{k_3^5 k_4}{k_2^6}$
The above equations can be integrated as

$$W = V + a_1 z + a_2,$$
$$P_{zz} - 6P_z^2 + (8C_1 - 1)P_z + iC_2 z + a3 = 0, \; P = V_z. \quad (34)$$

This last equation is nothing but the celebrated Painlevé II equation whose solutions has been extensively studied [8,11].

– *Case II.2.* $k_1 = 0, k_2 \neq 0, k_3 = 0, k_4 \neq 0, k_5 \neq 0$

- Reduced variables

$$z = \frac{k_5}{k_2}x_2,$$
$$\Lambda = \left(\frac{k_5}{k_2}\right)^{1/2}\left(\frac{k_2}{k_4}\lambda - x_1\right)$$

- Reduced fields

$$U(x_1, x_2) = \left(\frac{k_5}{k_2}\right)^{1/2} V(z) + \frac{k_5}{k_2}x_1,$$
$$\Omega(x_1, x_2) = \frac{k_5}{k_2}W(z) + i\frac{k_4}{k_2}x_2 x_1 + \frac{1}{k_2}\int k_6(x_1)dx_1.$$

- Reduced Eigenfunction

$$\psi(x_1, x_2) = \varphi(z, \Lambda)e^{\frac{1}{k_4}\int B(\lambda)d\lambda}.$$

- Reduced Lax pair

$$\varphi_{\Lambda\Lambda} + i\varphi_z + 2\varphi = 0,$$
$$\varphi_{zz} - 2iV_z\varphi_\Lambda + (W_z - \Lambda V_{zz})\varphi = 0. \quad (35)$$

- Reduced equation

$$V_{zz} = i\frac{k_2^{3/2}k_4}{k_5^{5/2}},$$
$$W_{zz} = 0. \quad (36)$$

– *Case II.3.* $k_1 = 0, k_2 = 0, k_3 \neq 0, k_4 \neq 0, k_5 \neq 0$

- Reduced variables

$$z = \left(\frac{k_5}{k_3}\right)^{1/3}x_1,$$
$$\Lambda = \frac{k_3^{1/3}k_5^{2/3}}{k_4}\left(\lambda - \frac{k_4}{k_3}x_2\right) + i\left(\frac{k_5}{k_3}\right)^{2/3}\frac{B_0}{k_4}$$

- Reduced fields

$$U(x_1, x_2) = \left(\frac{k_5}{2k_3}\right)^{1/3} V(z) + \frac{k_5}{k_3}x_2,$$
$$\Omega(x_1, x_2) = W(z) + i\frac{k_4}{2k_3}x_2^2 + \frac{B_0}{k_3}x_2.$$

where B_0 is a constant.

- Reduced Eigenfunction

$$\psi(x_1, x_2) = \varphi(z, \Lambda)e^{\frac{1}{k_4}\int B(\lambda)d\lambda}.$$

- Reduced Lax pair

$$\varphi_{\Lambda\Lambda} + 2i\varphi_z - 2\Lambda V_{zz}\varphi = 0,$$
$$\varphi_{zz} - i\varphi_\Lambda + V_z\varphi = 0.$$

Here, the constant "Λ" plays the role of the spectral paramenter.

- Reduced equation

$$V_{zz} = i\frac{k_3 k_4}{2k_5^2},$$
$$W_{zz} = 0. \quad (37)$$

4 Conclusions and brief comments

We have calculated the classical (point) symmetries of a nonlinear PDE and its corresponding two-component, nonisospectral Lax pair in 2 + 1 dimensions. The spectral parameter has been conveniently introduced as dependent variable obeying the nonisospectral condition. We have reduced both the equation and its Lax pair attending to different choices of the arbitrary functions and constants of integration present in the calculated symmetries. Of a total of 6 reductions, 2 of them happen to be knowledgeable equations in the literature. One is the KdV equation. From this result, we can say that $(2 + 1)$-BKP equation is a generalization of the KdV equation. The nonclassical version when $\xi_3 \neq 0$ has also been obtained (although it is not included in this manuscript), leading to the same set of symmetries.

The other interesting reduction is an equation showing multisoliton solutions. In this equation and its associated Lax pair, we have performed a second classical Lie symmetry analysis. In this case, the spectral parameter has been conveniently introduced in the reduction and it needs to be considered as an independent variable. Another 4 possible reductions are contemplated.

After the computation of the classical Lie symmetry results, the nonclassical symmetries for the nontrivial reduction I.1. studied in Sect. 3.1. Equation (24) and those of its corresponding Lax pair (25) have also been obtained, provided that $\xi_2 \neq 0$, leading us to equivalent results (not listed in this paper). Therefore, the classical and nonclassical symmetries of Eqs. (24) and (25) are equivalent. Nonetheless, this is not always the rule. Indeed, nonclassical symmetries are usually more general and contain the classical ones as a particular case.

As a future perspective, it would be desirable to be able to find ways of solving nonisospectral Lax pairs in $1 + 1$ dimensions, since the methods applicable are only applicable in the cases of isospectral ones.

Acknowledgements P. G. Estévez, J. D. Lejarreta and C. Sardón acknowledge partial financial support by research projects MAT2013-46308-C2-1-R (MINECO) and SA226013 (JCYL).

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