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Non-isospectral scattering problems and truncation for hierarchies: Burgers and dispersive water waves

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Abstract

Non-isospectral scattering problems have been proven useful for several reasons, amongst them the information that they provide about Painlevé truncation for entire hierarchies of integrable partial differential equations (PDEs). We show in this paper how our approach provides in a very straightforward manner truncation results for two hierarchies in $1+1$ dimensions, namely Burgers' hierarchy and the dispersive water wave hierarchy. Burgers' equation is well-known as a model for turbulence, and the dispersive water wave equations as a system governing shallow water waves. We also see how these results are easily extended to related hierarchies in $2+1$ dimensions. Our results for the $2+1$-dimensional Burgers' hierarchy, and for the $1+1$ and $2+1$-dimensional dispersive water wave hierarchies, are all new.

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1. Introduction

The truncation methods based on Painlevé analysis have been used now for several years to derive important properties of integrable partial differential equations (PDEs). Given a hierarchy of integrable PDEs it is then a natural question to ask how the truncation results can be extended from a single equation to the entire hierarchy. The first results in this direction are due to Weiss [1] and include, amongst other examples, truncation results for the Korteweg–de Vries (KdV) hierarchy. These results relied on the knowledge of certain properties of the hierarchies under consideration such as, for example, that of a modified hierarchy or of a Schwarzian formulation.

In a recent paper [2] we presented a new method of deriving truncation results for hierarchies of integrable PDEs which is considerably much simpler than that of Weiss and does not make use of the knowledge of the above-mentioned properties. In Ref. [2] we used the KdV hierarchy as an example to illustrate the effectiveness of the method, and we also observed that results for 1 + 1-dimensional hierarchies can be extended to related hierarchies in 2 + 1 dimensions.

The method introduced in Ref. [2] made use of PDEs associated to non-isospectral scattering problems, i.e., where the spectral parameter in the Lax pair is no longer constant but instead satisfies some differential equation [3–9]. In fact, our method has its starting point in our previous work [10–13] on such scattering problems, where we showed how a single equation can be used to characterize the entire corresponding non-isospectral hierarchy.

Thus in Ref. [2], instead of the KdV hierarchy

$$U_{t_{2n+1}} = \mathscr{R}^n U_x, \quad n = 0, 1, 2 \dots , \tag{1}$$

where $\mathscr{R}$ is the KdV recursion operator,

$$\mathscr{R} = \partial_x^2 + 4U + 2U_x\partial_x^{-1}, \quad \partial_x = \partial/\partial x , \tag{2}$$

we considered truncation for the equation

$$U_t = \mathscr{R} U_\tau . \tag{3}$$

Iteration of the results of this truncation then yields truncation result for the KdV hierarchy, or more generally for the hierarchy $U_{t_{2n+1}} = \mathscr{R}^n U_y$, depending on whether as base equation we take $U_{t_3} = \mathscr{R} U_X$ or $U_{t_3} = \mathscr{R} U_y$.

The aim of the present paper is to show the power of this new method through its application to other hierarchies based on physically interesting equations, namely Burgers' equation, well-known as a model for turbulence, and the dispersive water wave (DWW) system, which governs shallow water waves. In Section 2 we apply our method to Burgers' hierarchy, and in this way recover the results given in Ref. [14]; we also extend these results to obtain the truncation for the corresponding 2+1-dimensional hierarchy, which has not been given before. In Section 3 we consider the dispersive water wave hierarchy and derive truncation results for both the standard (isospectral) hierarchy [15–22], and also for the associated non-isospectral hierarchy [13]. Both sets of results—i.e., including even those obtained for the standard hierarchy—are new. Section 4 is devoted to conclusions and to a summary of our results.

2. Truncation for the 1 + 1 and 2 + 1-dimensional Burgers' hierarchies

In this section we consider Burgers' hierarchy, i.e.,

$$U_{t_{n+1}} = \mathscr{R}^n U_x = \partial_x L_n[U] = \partial_x (T[U])^n U, \quad n = 0, 1, \ldots , \tag{4}$$

where $T[U]$ and $\mathscr{R}$ are given by

$$T[U] = \partial_x + \tfrac{1}{2} U \tag{5}$$

and

$$\mathscr{R} = \partial_x T[U] \partial_x^{-1} , \tag{6}$$

respectively. The first non-trivial member of this hierarchy is the well-known Burgers' equation

$$U_{t_2} = (U_x + \tfrac{1}{2} U^2)_x . \tag{7}$$

As described in the Introduction, we therefore consider the equation $U_t = \mathscr{R} U_\tau$, which under the change of variables $U = u_x$ can be written in potential form as

$$u_t = u_{x\tau} + \tfrac{1}{2} u_x u_\tau . \tag{8}$$

We note that this equation was first given in Ref. [23]. We now consider the truncation for Eq. (8) in the standard Weiss–Tabor–Carnevale notation. It is easy to obtain that the truncated expansion is

$$u = 2 \log \varphi + v \tag{9}$$

with v a second solution of Eq. (8),

$$v_t = v_{x\tau} + \tfrac{1}{2} v_x v_\tau = (\partial_x + \tfrac{1}{2} v_x) v_\tau , \tag{10}$$

and with φ satisfying

$$\varphi_t = \varphi_{x\tau} + \tfrac{1}{2}(v_\tau \varphi_x + v_x \varphi_\tau) = (\partial_x + \tfrac{1}{2} v_x)\varphi_\tau + \tfrac{1}{2} v_\tau \varphi_x . \tag{11}$$

Eqs. (10) and (11) are the singular manifold equations for Eq. (8).

2.1. Truncation for Burgers' hierarchy

In order to obtain the truncation for Burgers' hierarchy we iterate on the above results, taking as starting point the singular manifold equations for (potential) Burgers' equation itself, which are readily obtained by making the reduction $\partial/\partial\tau = \partial/\partial x$ in the above results:

$$u = 2 \log \varphi + v \tag{12}$$

satisfies

$$u_{t_2} = u_{xx} + \tfrac{1}{2}u_x^2 \,, \tag{13}$$

if v satisfies the same equation

$$v_{t_2} = v_{xx} + \tfrac{1}{2}v_x^2 \,, \tag{14}$$

and if φ satisfies

$$\varphi_{t_2} = \varphi_{xx} + v_x\varphi_x \,. \tag{15}$$

We then set $t = t_{n+1}$ and $\tau = t_n$ in Eqs. (10) and (11) and iterate. From Eq. (10) we easily obtain

$$v_{t_{n+1}} = (\partial_x + \tfrac{1}{2}v_x)v_{t_n} = \cdots = (\partial_x + \tfrac{1}{2}v_x)^{n-1}v_{t_2} = (T[v_x])^n v_x \,. \tag{16}$$

Meanwhile, Eq. (11) gives

$$\varphi_{t_{n+1}} = (\partial_x + \tfrac{1}{2}v_x)\varphi_{t_n} + \tfrac{1}{2}v_{t_n}\varphi_x \,, \tag{17}$$

iteration of which leads to

$$\varphi_{t_{n+1}} = M_n'[v]\varphi \,, \tag{18}$$

where $M_n[v] = L_n[v_x]$ and $M_n'[v]$ is its Fréchet derivative. This we prove by induction. For $n = 1$ we have

$$\varphi_{t_2} = M_1'[v]\varphi = (\partial_x^2 + v_x\partial_x)\varphi = \varphi_{xx} + v_x\varphi_x \tag{19}$$

which is precisely Eq. (15) (we note that $M_1[v] = v_{xx} + \frac{1}{2}v_x^2$). We now assume that (18) is true for the t_n flow, that is, $\varphi_{t_n} = M_{n-1}'[v]\varphi$; then from (17)

$$\begin{aligned}\varphi_{t_{n+1}} &= (\partial_x + \tfrac{1}{2}v_x)\varphi_{t_n} + \tfrac{1}{2}v_{t_n}\varphi_x = T[v_x]M_{n-1}'[v]\varphi + \tfrac{1}{2}M_{n-1}[v]\partial_x\varphi \\ &= (T[v_x]M_{n-1}[v])'\varphi = M_n'[v]\varphi\end{aligned} \tag{20}$$

where we have used the fact that $M_n[v]$ satisfies the recursion relation $M_n[v] = T[v_x]M_{n-1}[v]$.

Thus we conclude that the singular manifold equations for the potential Burgers' hierarchy $u_{t_{n+1}} = M_n[u]$ are given by Eqs. (16) and (18). It then follows that the singular manifold equations for Burgers' hierarchy (4), corresponding to the truncation

$$U = 2\frac{\varphi_x}{\varphi} + V \,, \tag{21}$$

are

$$V_{t_{n+1}} = \partial_x L_n[V] \,, \tag{22}$$

i.e., that V is a second solution of (4), and

$$\varphi_{t_{n+1}} = L_n'[V]\varphi_x \,. \tag{23}$$

Thus we have recovered the results obtained in Ref. [14].

Setting $V=\varphi$ gives the special auto-Bäcklund transformation of Burgers' hierarchy

$$U=2\,\frac{\varphi_x}{\varphi}+\varphi\;, \tag{24}$$

and setting $V=0$ gives the linearization

$$U=2\,\frac{\varphi_x}{\varphi}\;, \tag{25}$$

onto

$$\varphi_{t_{n+1}}=\partial_x^{n+1}\varphi\;. \tag{26}$$

We note that in the case of Burgers' equation itself, the truncation results were originally obtained in Ref. [24] and the auto-Bäcklund transformation (24) in Ref. [25]. The linearization (25) is of course the Cole–Hopf transformation [26].

2.2. Truncation results for the 2 + 1-dimensional Burgers' hierarchy

We now consider the construction of the singular manifold equations for the 2 + 1-dimensional Burgers' hierarchy

$$U_{t_{n+1}}=\mathscr{R}^n U_y \tag{27}$$

which, using the same change of variables as before, $U=u_x$, we will consider in potential form

$$u_{t_{n+1}}=P_n[u]=(T[u_x])^n u_y\;. \tag{28}$$

The truncation

$$u=2\,\log\,\varphi+v \tag{29}$$

then gives, instead of Eq. (16),

$$v_{t_{n+1}}=P_n[v]=(T[v_x])^n v_y \tag{30}$$

since, for the 2 + 1-dimensional hierarchy, we have as base equation $v_{t_2}=T[v_x]v_y$.

Let us consider now the second of the singular manifold equations. For $n=1$ in (28) this second singular manifold equation is obtained from (11), with $t=t_2$ and $\tau=y$, as

$$\varphi_{t_2}=\varphi_{xy}+\tfrac{1}{2}v_x\varphi_y+\tfrac{1}{2}v_y\varphi_x\;, \tag{31}$$

which is precisely the linearization of $v_{t_2}=P_1[v]$,

$$\varphi_{t_2}=P_1'[v]\varphi\;, \tag{32}$$

where the Fréchet derivative of $P_n[v], P_n'[v]$, is defined as

$$P_n'[v]=\sum_{i,j}\frac{\partial P_n[v]}{\partial v_{i,j}}\partial_x^i\partial_y^j\;. \tag{33}$$

The same result holds for higher members of the hierarchy, i.e., instead of (18) we obtain that the second singular manifold equation is

$$\varphi_{t_{n+1}} = P'_n[v]\varphi \,, \tag{34}$$

a result that we again prove by induction. Assuming that (34) is true for the t_n flow, i.e., $\varphi_{t_n} = P'_{n-1}[v]\varphi$, we obtain from (11), with $t = t_{n+1}$ and $\tau = t_n$, that

$$\begin{aligned}\varphi_{t_{n+1}} &= (\partial_x + \tfrac{1}{2}v_x)\varphi_{t_n} + \tfrac{1}{2}v_{t_n}\varphi_x = T[v_x]P'_{n-1}[v]\varphi + \tfrac{1}{2}P_{n-1}[v]\partial_x\varphi \\ &= (T[v_x]P_{n-1}[v])'\varphi = P'_n[v]\varphi \,.\end{aligned} \tag{35}$$

We thus obtain that Eqs. (30) and (34) are precisely the singular manifold equations for the 2 + 1-dimensional Burgers' hierarchy.

We note in addition that we also have, setting $\varphi = v_x$, the special auto-Bäcklund transformation of the hierarchy (28),

$$u = 2\,\log(v_x) + v \,, \tag{36}$$

and, setting $v = 0$, the linearization

$$u = 2\,\log\varphi \tag{37}$$

of the hierarchy (28) onto

$$\varphi_{t_{n+1}} = \partial_x^n\varphi_y \,. \tag{38}$$

The first of these is the $2+1$-dimensional generalization of the special auto-Bäcklund transformation (24) of Burgers' hierarchy, and the second is the $2+1$-dimensional generalization of the Cole–Hopf transformation, for the entire hierarchy.

3. Truncation for the 1 + 1 and 2 + 1-dimensional DWW hierarchies

This section is concerned with the derivation of truncation results for the DWW hierarchy [15–22] and the corresponding $2+1$-dimensional non-isospectral hierarchy. Truncation results for these hierarchies have not been given before.

We begin with the standard DWW hierarchy, which we take here in the form

$$\mathbf{W}_{t_n} = \mathcal{R}^n\mathbf{W}_x, \quad n = 0, 1, \ldots \,, \tag{39}$$

where $\mathbf{W} = (W, V)^{\mathrm{T}}$, $\mathcal{R} = B_2B_1^{-1}$ is the recursion operator, and B_2 and B_1 are two of the Hamiltonian operators of the DWW hierarchy, given by

$$B_2 = -\tfrac{1}{2}\begin{pmatrix} 2\partial_x & \partial_x W \\ W\partial_x & 2\partial_x^3 + 2V\partial_x + V_x \end{pmatrix}, \tag{40}$$

and

$$B_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}. \tag{41}$$

In terms of the Hamiltonian operators B_2 and B_1 the hierarchy (39) can be expressed in the form

$$\mathbf{W}_{t_n} = B_1 \mathbf{L}_{n+1}[\mathbf{W}] = B_2 \mathbf{L}_n[\mathbf{W}] , \tag{42}$$

where $\mathbf{L}_n = (M_n, N_n)^{\mathrm{T}}$, $\mathbf{L}_0 = (0, -2)^{\mathrm{T}}$ and $\mathbf{L}_1 = (V, W)^{\mathrm{T}}$.

The first step in our method consists of obtaining truncation results for an equation of the form

$$\mathbf{W}_t = \mathscr{R}\mathbf{W}_\tau . \tag{43}$$

Here we consider the slight generalization

$$U_{xt} + V_\tau + \tfrac{1}{2}(U_x U_\tau)_x + g(\tau, t) = 0 , \tag{44}$$

$$V_t + U_{xxx\tau} + U_{x\tau} V + \tfrac{1}{2}(U_x V_\tau + U_\tau V_x) = 0 , \tag{45}$$

where we have set $W = U_x$, in order to write the system in local form.

The truncation for the system (44), (45) was performed in Ref. [27]. We summarize here the results: U and V given by

$$U = 2 \log Z + u , \tag{46}$$

$$V = 2(Z^{-1} + BZ)_x - 4B , \tag{47}$$

satisfy the system (44), (45), where Z satisfies the Riccati system

$$Z_x = 1 - AZ - BZ^2 , \tag{48}$$

$$Z_t = -C^t + (AC^t + C^t_x)Z - (D^t - BC^t)Z^2 , \tag{49}$$

$$Z_\tau = -C^\tau + (AC^\tau + C^\tau_x)Z - (D^\tau - BC^\tau)Z^2 , \tag{50}$$

and the coefficients A, B, C^t, D^t, C^τ and D^τ are given by

$$A = -\lambda + \tfrac{1}{2}u_x , \tag{51}$$

$$B = -\tfrac{1}{4}(v - u_{xx}) , \tag{52}$$

$$C^t = -\lambda C^\tau + \tfrac{1}{2}u_\tau , \tag{53}$$

$$D^t = -\lambda D^\tau - \tfrac{1}{4}(v - u_{xx})_\tau , \tag{54}$$

$\lambda(\tau, t)$ is a function of integration that satisfies the relation $g + 2(\lambda_t + \lambda\lambda_\tau) = 0$ and u and v are solutions of the system (44), (45).

Since we have the relations

$$C^t = \Gamma C^\tau + \tilde{C}, \quad D^t = \Gamma D^\tau + \tilde{D}, \quad \Gamma_x = 0 , \tag{55}$$

we can write the Riccati system (48)–(50) in the form

$$Z_x = 1 - AZ - BZ^2 , \tag{56}$$

$$Z_t = \Gamma Z_\tau - \tilde{C} + (A\tilde{C} + \tilde{C}_x)Z - (\tilde{D} - B\tilde{C})Z^2 , \tag{57}$$

where $\tilde{C} = \frac{1}{2}u_\tau$, $\tilde{D} = -\frac{1}{4}(v - u_{xx})_\tau$ and $\Gamma = -\lambda$. Imposing now the additional condition $g = 0$, the linearization of the above Riccati system provides the Lax pair for the system (43), together with the non-isospectral condition $\lambda_t = -\lambda\lambda_\tau$, and the truncation (46), (47) is precisely its Darboux transformation.

3.1. Truncation for the DWW hierarchy

We now consider iterating to obtain truncation results for the standard DWW hierarchy (39). We therefore need, as a starting point for the iteration, the truncation for the reduction $\partial/\partial_\tau = \partial/\partial_x$ of the system (43) with $t = t_1$, i.e.,

$$U_{xt_1} + \left(V + \frac{1}{2}\,U_x^2\right)_x = 0\,, \tag{58}$$

$$V_{t_1} + (U_{xxx} + U_x V)_x = 0\,. \tag{59}$$

This system is the classical Boussinesq system. Truncation for the classical Boussinesq system has been considered by several different authors, e.g. [22,28,27]; here we will adopt the approach used in Ref. [27].

Under this reduction we have $C^\tau = -1$ and $D^\tau = 0$, so that

$$U = 2\,\log Z + u\,, \tag{60}$$

$$V = 2(Z^{-1} + BZ)_x - 4B\,, \tag{61}$$

satisfy the system (58), (59), where Z satisfies the Riccati system

$$Z_x = 1 - AZ - BZ^2\,, \tag{62}$$

$$Z_{t_1} = -C^{t_1} + (AC^{t_1} + C^{t_1}_x)Z - (D^{t_1} - BC^{t_1})Z^2\,, \tag{63}$$

with coefficients

$$A = -\lambda + \tfrac{1}{2}u_x\,, \tag{64}$$

$$B = -\tfrac{1}{4}(v - u_{xx})\,, \tag{65}$$

$$C^{t_1} = \lambda + \tfrac{1}{2}u_x\,, \tag{66}$$

$$D^{t_1} = -\tfrac{1}{4}(v - u_{xx})_x\,, \tag{67}$$

and where λ is now constant.

We now turn to the iteration process. Truncation for the $1+1$-dimensional DWW hierarchy (39) gives the Darboux transformation

$$W = 2(\log Z)_x + w\,, \tag{68}$$

$$V = 2(Z^{-1} + BZ)_x - 4B\,, \tag{69}$$

with $\mathbf{w} = (w, v)^{\mathrm{T}} = (u_x, v)^{\mathrm{T}}$ being a second solution of (39), together with the Lax pair obtained from the linearization of the Riccati system for the t_n flow,

$$Z_x = 1 - AZ - BZ^2 , \tag{70}$$

$$Z_{t_n} = -C^{t_n} + (AC^{t_n} + C_x^{t_n})Z - (D^{t_n} - BC^{t_n})Z^2 . \tag{71}$$

In order to obtain the coefficients of the above Riccati system, it only remains to iterate in Eqs. (53) and (54). We note that the relation $\lambda_{t_n} = -\lambda\lambda_{t_{n-1}}$ gives that λ is a constant for the entire hierarchy.

Let us consider first Eq. (53). We set $t = t_n$ and $\tau = t_{n-1}$ and iterate to obtain:

$$C^{t_n} = -\lambda C^{t_{n-1}} + \tfrac{1}{2} u_{t_{n-1}} \tag{72}$$

$$= -\lambda(-\lambda C^{t_{n-2}} + \tfrac{1}{2} u_{t_{n-2}}) + \tfrac{1}{2} u_{t_{n-1}} \tag{73}$$

$$\vdots$$

$$= (-\lambda)^{n-1} C^{t_1} + \frac{1}{2} \sum_{k=2}^{n} (-\lambda)^{n-k} u_{t_{k-1}} . \tag{74}$$

Iterating Eq. (54) we obtain, similarly,

$$D^{t_n} = -\lambda D^{t_{n-1}} - \tfrac{1}{4}(v - u_{xx})_{t_{n-1}} \tag{75}$$

$$= (-\lambda)^{n-1} D^{t_1} - \frac{1}{4} \sum_{k=2}^{n} (-\lambda)^{n-k} [v_{t_{k-1}} - (u_{xx})_{t_{k-1}}] . \tag{76}$$

From (42) we obtain $u_{t_{k-1}} = N_k[\mathbf{w}]$ and $v_{t_{k-1}} = \partial_x M_k[\mathbf{w}]$. We also have, from Eqs. (66) and (67),

$$C^{t_1} = \lambda + \tfrac{1}{2} w , \tag{77}$$

$$D^{t_1} = -\tfrac{1}{4}(v_x - w_{xx}) . \tag{78}$$

Substituting into (74) then gives

$$C^{t_n} = (-\lambda)^{n-1} C^{t_1} + \frac{1}{2} \sum_{k=2}^{n} (-\lambda)^{n-k} u_{t_{k-1}} \tag{79}$$

$$= (-\lambda)^{n-1} (\lambda + \tfrac{1}{2} w) + \frac{1}{2} \sum_{k=2}^{n} (-\lambda)^{n-k} N_k[\mathbf{w}] \tag{80}$$

$$= \frac{1}{2} \sum_{k=0}^{n} (-\lambda)^{n-k} N_k[\mathbf{w}] , \tag{81}$$

where we have used that $N_0[\mathbf{w}] = -2$ and $N_1[\mathbf{w}] = w$.

Substituting into Eq. (76) we obtain

$$D^{t_n} = (-\lambda)^{n-1} D^{t_1} - \frac{1}{4}\sum_{k=2}^{n}(-\lambda)^{n-k}[v_{t_{k-1}} - (u_{xx})_{t_{k-1}}] \tag{82}$$

$$= -\tfrac{1}{4}(-\lambda)^{n-1}(v_x - w_{xx}) - \frac{1}{4}\sum_{k=2}^{n}(-\lambda)^{n-k}[\partial_x M_k[\mathbf{w}] - \partial_x^2 N_k[\mathbf{w}]] \tag{83}$$

$$= -\frac{1}{4}\sum_{k=1}^{n}(-\lambda)^{n-k}[\partial_x M_k[\mathbf{w}] - \partial_x^2 N_k[\mathbf{w}]]\,, \tag{84}$$

where we now have also used that $M_1[\mathbf{w}] = v$.

Thus we obtain truncation results for entire DWW hierarchy. We see that the result of the truncation is the recovery of the Lax pair and Darboux transformation for every member of the DWW hierarchy. These results are new.

3.2. Truncation for the non-isospectral DWW hierarchy

We now consider the derivation of truncation results for the 2 + 1-dimensional DWW hierarchy

$$\mathbf{W}_{t_n} = \mathscr{R}^n \mathbf{W}_y, \quad n = 0, 1, \ldots\ . \tag{85}$$

Using the same change of variables as before, that is, $\mathbf{W} = (U_x, V)^{\mathrm{T}}$, we introduce the quantities Q_k and P_k defined by the relation $\mathscr{R}^{k-1}(U_{xy}, V_y)^{\mathrm{T}} = (Q_k, P_k)^{\mathrm{T}}$, and take the hierarchy (85) in the form

$$\begin{pmatrix} U_x \\ V \end{pmatrix}_{t_n} = \begin{pmatrix} Q_{n+1} \\ P_{n+1} \end{pmatrix}. \tag{86}$$

The result of the truncation is that

$$U = 2\log Z + u\,, \tag{87}$$

$$V = 2(Z^{-1} + BZ)_x - 4B\,, \tag{88}$$

is a solution of (86), where as before

$$A = -\lambda + \tfrac{1}{2}u_x\,, \tag{89}$$

$$B = -\tfrac{1}{4}(v - u_{xx})\,, \tag{90}$$

and (u, v) is a second solution of (86). Here Z satisfies the Riccati system

$$Z_x = 1 - AZ - BZ^2\,, \tag{91}$$

$$Z_{t_n} = -C^{t_n} + (AC^{t_n} + C_x^{t_n})Z - (D^{t_n} - BC^{t_n})Z^2\,, \tag{92}$$

$$Z_y = -C^y + (AC^y + C_x^y)Z - (D^y - BC^y)Z^2\,. \tag{93}$$

We now need to obtain, as for the standard DWW hierarchy, the expressions for the coefficients C^{t_n} and D^{t_n}, as well as the equation satisfied by the spectral parameter.

This last is obtained by iteration as

$$\lambda_{t_n} = -\lambda\lambda_{t_{n-1}} = \cdots = (-\lambda)^{n-1}\lambda_{t_1} = (-\lambda)^n \lambda_y \,, \tag{94}$$

which then gives the non-isospectral condition for the t_n flow (86).

We now turn to the derivation of the expressions for the coefficients in the Riccati system. Iterating in Eq. (53) we obtain

$$\begin{aligned} C^{t_n} &= (-\lambda)^{n-1}C^{t_1} + \frac{1}{2}\sum_{k=2}^{n}(-\lambda)^{n-k}u_{t_{k-1}} \\ &= (-\lambda)^{n-1}\left(-\lambda C^y + \frac{1}{2}u_y\right) + \frac{1}{2}\sum_{k=2}^{n}(-\lambda)^{n-k}u_{t_{k-1}} \\ &= (-\lambda)^n C^y + \frac{1}{2}\sum_{k=1}^{n}(-\lambda)^{n-k}u_{t_{k-1}} \,, \end{aligned} \tag{95}$$

whereas iteration in Eq. (54) yields

$$\begin{aligned} D^{t_n} &= (-\lambda)^{n-1}D^{t_1} - \frac{1}{4}\sum_{k=2}^{n}(-\lambda)^{n-k}[v_{t_{k-1}} - (u_{xx})_{t_{k-1}}] \\ &= (-\lambda)^{n-1}[-\lambda D^y - \tfrac{1}{4}(v_x - w_{xx})_y] - \frac{1}{4}\sum_{k=2}^{n}(-\lambda)^{n-k}[v_{t_{k-1}} - (u_{xx})_{t_{k-1}}] \\ &= (-\lambda)^n D^y - \frac{1}{4}\sum_{k=1}^{n}(-\lambda)^{n-k}[v_{t_{k-1}} - (u_{xx})_{t_{k-1}}] \,, \end{aligned} \tag{96}$$

where in these last we have used Eqs. (53) and (54) with $t = t_1$ and $\tau = y$, and also that $u_{t_0} = u_y$ and $v_{t_0} = v_y$.

Moreover, using the fact that $u_{t_{k-1}} = \partial_x^{-1}Q_k$ and $v_{t_{k-1}} = P_k$, where now $Q_k = Q_k(u, v)$ and $P_k = P_k(u, v)$, we can write the above expressions for C^{t_n} and D^{t_n} in the form

$$C^{t_n} = (-\lambda)^n C^y + \frac{1}{2}\sum_{k=1}^{n}(-\lambda)^{n-k}\partial_x^{-1}Q_k \,, \tag{97}$$

$$D^{t_n} = (-\lambda)^n D^y - \frac{1}{4}\sum_{k=1}^{n}(-\lambda)^{n-k}[P_k - \partial_x Q_k] \,. \tag{98}$$

We note here that, in the reduction to the standard DWW hierarchy, we have $C^y = -1$, $D^y = 0$, $Q_k = \partial_x N_k$ and $P_k = \partial_x M_k$, and thus we recover the results in Section 3.1.

We observe that Eqs. (97) and (98) are of the form

$$C^{t_n} = \hat{\Gamma} C^y + \hat{C}, \quad D^{t_n} = \hat{\Gamma} D^y + \hat{D}, \quad \hat{\Gamma}_x = 0 \,, \tag{99}$$

where $\hat{\Gamma} = (-\lambda)^n$ and

$$\hat{C} = \frac{1}{2}\sum_{k=1}^{n}(-\lambda)^{n-k}\partial_x^{-1}Q_k \,, \tag{100}$$

$$\hat{D} = -\frac{1}{4}\sum_{k=1}^{n}(-\lambda)^{n-k}[P_k - \partial_x Q_k] \,, \tag{101}$$

and so we can replace the Riccati system (91), (92), (93) with

$$Z_x = 1 - AZ - BZ^2 \,, \tag{102}$$

$$Z_{t_n} = \hat{\Gamma} Z_y - \hat{C} + (A\hat{C} + \hat{C}_x)Z - (\hat{D} - B\hat{C})Z^2 \,. \tag{103}$$

The linearization of this last, together with the non-isospectral condition $\lambda_{t_n} = (-\lambda)^n \lambda_y$, provides the Lax pair for the hierarchy (86); its Darboux transformation is given by (87), (88). Thus we obtain truncation results for the 2 + 1-dimensional hierarchy; in fact, we see that the result of the truncation is the recovery of the Lax pair and Darboux transformation for every member of this hierarchy.

4. Conclusions

We have used our new approach to obtain truncation results for hierarchies in order to derive such for the $1+1$ and $2+1$-dimensional versions of both Burgers' and the dispersive water wave hierarchies. Our results for the $2+1$-dimensional Burgers' hierarchy, and for the $1+1$ and $2+1$-dimensional dispersive water wave hierarchies, are all new. We note that for these last the singular manifold equation can be obtained using a gauge transformation (see Ref. [27]); however, it is the derivation, from truncation, of the Lax pair and Darboux transformation for the entire hierarchy, as given here, that is of most interest. We also note that our approach is based on the connection between non-isospectral scattering problems and hierarchies of integrable PDEs, obtained by iteration of the former [11]. That is, we iterate on (the generalized form of) the truncation for equations of the form $\mathbf{V}_t = R\mathbf{V}_\tau$, for some recursion operator R; iteration on g in (44), (45) is also of course possible.

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