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# Symmetry reductions of a $2+1$ Lax pair

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**Abstract**

In this Letter we present the reductions arising from the classical Lie symmetries of a Lax pair in $2+1$ dimensions. We obtain several interesting reductions and prove that, by analyzing not only a PDE but also its associated linear problem, it is possible to obtain the reduction of the PDE together with the reduced Lax pair. Specially relevant is the fact that the spectral parameter in $1+1$ dimensions appears as a natural consequence of the reduction itself and is related to the symmetry of the $2+1$ eigenfunction.


*Keywords:* Exactly solvable and integrable systems

## 1. Introduction

The identification of the Lie symmetries of a given partial differential equation (PDE) is an instrument of primary importance in order to solve such an equation [1]. A standard method for finding solutions of PDEs is that of reduction using Lie symmetries: each Lie symmetry allows a reduction of the PDE to a new equation with the number of independent variables reduced by one [2,3]. In a certain way this procedure gives rise to the ARS conjecture [4] which establishes that a PDE is integrable in the Painlevé sense [5] if all its reductions pass the Painlevé test [6]. This means that solutions of a PDE can be achieved by solving its reductions to ordinary differential equations (ODE). Classical [1] and nonclassical [2,3] Lie symmetries are the usual way for identifying the reductions.

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Nevertheless, let us recall that there are some methods for solving PDEs that are more effective in $2+1$ than in $1+1$ dimensions [7]. A good example of this is the following equation in $2+1$ dimensions

$$\left[h_{xxz} - \frac{3}{4}\left(\frac{h_{xz}^2}{h_z}\right) + 3h_x h_z\right]_x = h_{yz}, \tag{1}$$

which some of us have studied in a recent paper [7] proving that the singular manifold method [6] is a very effective method for solving the equation. It is quite straightforward to determine the associated linear problem through this method. In fact, for (1) we obtained the following Lax pair:

$$-\psi_y + \psi_{xxx} + 3h_x\psi_x + \frac{3}{2}h_{xx}\psi = 0, \qquad 2h_z\psi_{xz} - h_{xz}\psi_z + 2h_z^2\psi = 0. \tag{2}$$

Notice that there are a lot of papers related with the reduction of $2+1$ equations through the classical Lie method but the application of this method to the Lax pair is much less frequent [8]. Nevertheless we consider that for integrable equations, it is of primary importance to determine the reduction, not only of the equation, but of the Lax pair. Actually the reduction process should introduce a spectral parameter that is absolutely essential in $1+1$ dimensions. Therefore, our plan in this Letter is to a certain extent exactly the opposite of the usual approach: we try to obtain $1+1$ spectral problems arising from a $2+1$ Lax pair. Once we have solved the problem in $2+1$ dimensions in [7], in the sense that we have determined its Lax pair, we shall identify the classical symmetries of the Lax pair [8]. This is done in Section 2. In Section 3 we use these symmetries to obtain a reduced Lax pair in $1+1$ dimensions whose compatibility condition should be a reduction of (1). Actually, there are five possible reductions. Two of them yield linear equations that can be easily integrated. The other three reductions yield $1+1$ spectral problems that include, as particular cases, well-known equations such as the modified Korteveg–de Vries, Drinfel'd–Sokolov or Ermakov–Pinney equations. It is interesting to note that each of these reductions yields respectively *two, three and fourth order spectral problems which exhibit a spectral parameter as a natural output of the Lie method*. We close with a section of conclusions.

## 2. Classical symmetries

In order to apply the classical Lie method to the system of PDEs (2) with three independent variables and two fields, we consider the one-parameter Lie group of infinitesimal transformations in $x, y, z, h, \psi$, given by:

$$\begin{aligned}
&x' = x + \varepsilon\xi_1(x,y,z,h,\psi) + O(\varepsilon^2), \qquad y' = y + \varepsilon\xi_2(x,y,z,h,\psi) + O(\varepsilon^2),\\
&z' = z + \varepsilon\xi_3(x,y,z,h,\psi) + O(\varepsilon^2), \qquad h' = h + \varepsilon\phi_1(x,y,z,h,\psi) + O(\varepsilon^2),\\
&\psi' = \psi + \varepsilon\phi_2(x,y,z,h,\psi) + O(\varepsilon^2),
\end{aligned} \tag{3}$$

where $\varepsilon$ is the group parameter. It is therefore necessary that this one transformation leaves the set of solutions of (2) invariant. This yields an overdetermined linear system of equations for the infinitesimals $\xi_1(x,y,z,h,\psi)$, $\xi_2(x,y,z,h,\psi)$, $\xi_3(x,y,z,h,\psi)$, $\phi_1(x,y,z,h,\psi)$ and $\phi_2(x,y,z,h,\psi)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form:

$$X = \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial y} + \xi_3\frac{\partial}{\partial z} + \phi_1\frac{\partial}{\partial h} + \phi_2\frac{\partial}{\partial\psi}. \tag{4}$$

By applying the classical method [1] to the system of PDEs (2), we obtain the following system of determining equations (we have used MACSYMA and MAPLE independently to handle the calculations):

$$0 = \frac{\partial\xi_1}{\partial z} = \frac{\partial\xi_1}{\partial h} = \frac{\partial\xi_1}{\partial\psi} = \frac{\partial\xi_2}{\partial x} = \frac{\partial\xi_2}{\partial z} = \frac{\partial\xi_2}{\partial h} = \frac{\partial\xi_2}{\partial\psi} = \frac{\partial\xi_3}{\partial x} = \frac{\partial\xi_3}{\partial y} = \frac{\partial\xi_3}{\partial h} = \frac{\partial\xi_3}{\partial\psi},$$

$$0=\frac{\partial\phi_1}{\partial z}=\frac{\partial^2\phi_1}{\partial h^2}=\frac{\partial\phi_1}{\partial\psi}=\frac{\partial\phi_2}{\partial z}=\frac{\partial\phi_2}{\partial h}=\frac{\partial^2\phi_2}{\partial\psi^2},$$
$$0=\frac{\partial^2\phi_2}{\partial x\,\partial\psi}-\frac{\partial^2\xi_1}{\partial x^2}=\frac{\partial^2\phi_1}{\partial x\,\partial h}-2\frac{\partial^2\phi_2}{\partial x\,\partial\psi}=\frac{\partial\xi_1}{\partial x}+\frac{\partial\phi_1}{\partial h}=3\frac{\partial\xi_1}{\partial x}-\frac{\partial\xi_2}{\partial y},$$
$$0=\phi_2+\psi\left(\frac{\partial\phi_1}{\partial h}-\frac{\partial\phi_2}{\partial\psi}+\frac{\partial\xi_1}{\partial x}\right)=\frac{\partial\xi_1}{\partial y}-\frac{\partial^3\xi_1}{\partial x^3}+3\frac{\partial^3\phi_2}{\partial x^2\,\partial\psi}+3\frac{\partial\phi_1}{\partial x},$$
$$0=2\frac{\partial\phi_2}{\partial x}+\psi\left(2\frac{\partial^2\phi_1}{\partial x\,\partial h}-\frac{\partial^2\xi_1}{\partial x^2}\right)=-2\frac{\partial\phi_2}{\partial y}+2\frac{\partial^3\phi_2}{\partial x^3}+3\psi\frac{\partial^2\phi_1}{\partial x^2},$$

whose solution is:

$$\xi_1=-2\frac{dA_1(y)}{dy}x-\frac{3}{2}A_2(y),\qquad \xi_2=-6A_1(y),\qquad \xi_3=\beta(z),$$
$$\phi_1=\frac{1}{3}\frac{d^2A_1(y)}{dy^2}x^2+\frac{1}{2}\frac{dA_2(y)}{dy}x+A_3(y)+2\frac{dA_1(y)}{dy}h,\qquad \phi_2=\left(\lambda+\frac{dA_1(y)}{dy}\right)\psi. \tag{5}$$

Note that the corresponding Lie symmetry algebra depends on three $A_1$, $A_2$ and $A_3$ arbitrary functions of $y$ and an arbitrary function $\beta(z)$ of $z$. The only constant that appears in (5) is $\lambda$ that, as we shall show in the next section, plays the role of the spectral parameter in the $1+1$ reductions of (2).

Having determined the infinitesimals of (3) in (5), the symmetry variables are found by solving the invariant surface conditions

$$\Phi_1\equiv\xi_1\frac{\partial h}{\partial x}+\xi_2\frac{\partial h}{\partial y}+\xi_3\frac{\partial h}{\partial z}-\phi_1=0,\qquad \Phi_2\equiv\xi_1\frac{\partial\psi}{\partial x}+\xi_2\frac{\partial\psi}{\partial y}+\xi_3\frac{\partial\psi}{\partial z}-\phi_2=0 \tag{6}$$

or the corresponding characteristic equations

$$\frac{-dx}{2\frac{dA_1}{dy}x+\frac{3}{2}A_2}=\frac{-dy}{6A_1}=\frac{dz}{\beta}=\frac{dh}{\frac{1}{3}\frac{d^2A_1}{dy^2}x^2+\frac{1}{2}\frac{dA_2}{dy}x+A_3+2\frac{dA_1}{dy}h}=\frac{d\psi}{(\lambda+\frac{dA_1}{dy})\psi}. \tag{7}$$

In the next section we solve (7) for the different possibilities.

## 3. Reductions

There are five independent reductions that we determine in the following way:

**Case 1.** $A_1\neq 0$, $\beta\neq 0$.

Integration of (7) provides the reduced variables:

$$z_1=\frac{x}{A_1^{1/3}}-\frac{1}{4}\int\frac{A_2}{A_1^{4/3}}\,dy,\qquad z_2=6\int\frac{1}{\beta}\,dz+\int\frac{1}{A_1}\,dy \tag{8}$$

and the following reduction for the fields

$$\psi(x,y,z)=\frac{e^{-\frac{\lambda}{6}\int\frac{dy}{A_1}}}{A_1^{1/6}}G(z_1,z_2), \tag{9}$$

$$h(x,y,z)=\frac{1}{A_1^{1/3}}U(z_1,z_2)-\frac{1}{6A_1^{1/3}}\int A_1^{1/3}M(z_1,y)\,dy, \tag{10}$$

where $M(z_1, y)$ is the function

$$M(z_1, y) = \left(\frac{d^2A_1}{dy^2}\frac{1}{3A_1^{1/3}}\right)z_1^2 + \left(\frac{dA_2}{dy}\frac{1}{2A_1^{2/3}} + \frac{d^2A_1}{dy^2}\frac{1}{6A_1^{1/3}}\int\frac{A_2}{A_1^{4/3}}\,dy\right)z_1 + \frac{d^2A_1}{dy^2}\frac{1}{48A_1^{1/3}}\left(\int\frac{A_2}{A_1^{4/3}}\,dy\right)^2 + \frac{dA_2}{dy}\frac{1}{8A_1^{2/3}}\int\frac{A_2}{A_1^{4/3}}\,dy + \frac{A_3}{A_1}. \tag{11}$$

Substitution of the reduction ansatz (8)–(10) in (2) gives us:

$$0 = G_{z_1z_1z_1} - G_{z_2} + 3U_{z_1}G_{z_1} + \frac{3}{2}U_{z_1z_1}G + \frac{\lambda}{6}G, \tag{12}$$

$$0 = 2U_{z_2}G_{z_2z_1} - U_{z_2z_1}G_{z_2} + 2U_{z_2}^2G. \tag{13}$$

Solving (12) for $G_{z_2}$ and substituting in (13), we obtain *the fourth order spectral problem*:

$$G_{z_2} = G_{z_1z_1z_1} + 3U_{z_1}G_{z_1} + \frac{3}{2}U_{z_1z_1}G + \frac{\lambda}{6}G, \tag{14}$$

$$U_{z_2}G_{z_1z_1z_1z_1} = \frac{1}{2}U_{z_2z_1}G_{z_1z_1z_1} - 3U_{z_1}U_{z_2}G_{z_1z_1} + \left(\frac{3}{2}U_{z_1}U_{z_2z_1} - \frac{\lambda}{6}U_{z_2} - \frac{9}{2}U_{z_1z_1}U_{z_2}\right)G_{z_1} + \left(\frac{\lambda}{12}U_{z_2z_1} - U_{z_2}^2 - \frac{3}{2}U_{z_1z_1z_1}U_{z_2} + \frac{3}{4}U_{z_1z_1}U_{z_2z_1}\right)G, \tag{15}$$

whose compatibility condition gives us the $1+1$ equation

$$\left(U_{z_2z_1z_1} - \frac{3}{4}\frac{U_{z_2z_1}^2}{U_{z_2}} + 3U_{z_2}U_{z_1}\right)_{z_1} = U_{z_2z_2} \tag{16}$$

that can be written as the following nonlocal KdV equation [9], proposed by Drinfel'd and Sokolov [10].

$$-P_{z_2} + P_{z_1z_1z_1} + 2PV_{z_1} + 4VP_{z_1} = 0, \qquad V_{z_2} = -\frac{3}{4}\left(P^2\right)_{z_1}, \tag{17}$$

where we have done the change

$$U_{z_1} = \frac{4}{3}V, \qquad U_{z_2} = -P^2. \tag{18}$$

**Case 2.** $A_1 \neq 0$, $\beta = 0$.

Integration of (7) provides the reduced variables:

$$z_1 = \frac{x}{A_1^{1/3}} - \frac{1}{4}\int\frac{A_2}{A_1^{4/3}}\,dy, \qquad z_2 = z \tag{19}$$

and the reduction for the fields is exactly the same as in Case 1. By substituting the reduction ansatz in (2) we obtain *the third order spectral problem*:

$$0 = G_{z_1z_1z_1} + 3U_{z_1}G_{z_1} + \frac{3}{2}U_{z_1z_1}G + \frac{\lambda}{6}G, \tag{20}$$

$$0 = 2U_{z_2}G_{z_2z_1} - U_{z_2z_1}G_{z_2} + 2U_{z_2}^2G, \tag{21}$$

whose compatibility gives us the $1+1$ equation

$$\left(U_{z_2z_1z_1} - \frac{3}{4}\frac{U_{z_2z_1}^2}{U_{z_2}} + 3U_{z_2}U_{z_1}\right)_{z_1} = 0. \tag{22}$$

An alternative form of the above equation arises from the following definitions

$$U_{z_1} = \frac{4}{3}V, \qquad U_{z_2} = -P^2, \tag{23}$$

which allow us to write (22) as the system

$$PP_{z_1z_1} - \frac{P_{z_1}^2}{2} + 2P^2V + F(z_2) = 0, \qquad V_{z_2} = -\frac{3}{4}\left(P^2\right)_{z_1}, \tag{24}$$

where $F(z_2)$ is an arbitrary function. Eq. (24) is the Ermakov–Pinney equation [11]. It has been proved in [12] that this equation is related through a reciprocal transformation to the Degasperis–Procesi equation [13].

**Case 3.** $A_1 = 0$, $A_2 \neq 0$, $\beta \neq 0$.

Integration of (7) yields the following reduction:

$$z_1 = x + \frac{3}{2}A_2\int\frac{1}{\beta}\,dz, \qquad z_2 = y, \tag{25}$$

$$\psi(x,y,z) = e^{(-\frac{2\lambda}{3A_2})x}G(z_1,z_2), \tag{26}$$

$$h(x,y,z) = U(z_1,z_2) - \frac{1}{6}\frac{\frac{d^2A_2}{dy^2}}{A_2}x^2 - \frac{2A_3}{3A_2}x. \tag{27}$$

Substitution of the reduction ansatz (25)–(27) in (2) gives us:

$$\begin{aligned}0 = {} & G_{z_1z_1z_1} - G_{z_2} - \frac{2\lambda}{A_2}G_{z_1z_1} + \left(3U_{z_1} + \frac{4\lambda^2}{3A_2^2} - \frac{2A_3}{A_2} - \frac{dA_2}{dz_2}\frac{z_1}{A_2}\right)G_{z_1} \\ & + \left(\frac{3}{2}U_{z_1z_1} - \frac{2\lambda}{A_2}U_{z_1} + \frac{4\lambda A_3}{3A_2^2} - \frac{8\lambda^3}{27A_2^3} - \frac{1}{2A_2}\frac{dA_2}{dz_2}\right)G,\end{aligned} \tag{28}$$

$$0 = U_{z_1}G_{z_1z_1} - \left(\frac{1}{2}U_{z_1z_1} + \frac{2\lambda}{3A_2}U_{z_1}\right)G_{z_1} + U_{z_1}^2G. \tag{29}$$

Solving (29) for $G_{z_1z_1}$ and substituting in (28), we obtain *the second order linear system*:

$$G_{z_1z_1} = \left(\frac{2\lambda}{3A_2} + \frac{U_{z_1z_1}}{2U_{z_1}}\right)G_{z_1} - U_{z_1}G, \tag{30}$$

$$\begin{aligned}G_{z_2} = {} & \left(\frac{4\lambda^2}{9A_2^2} - \frac{U_{z_1z_1}^2}{4U_{z_1}^2} - \frac{\lambda}{3A_2}\frac{U_{z_1z_1}}{U_{z_1}} - \frac{z_1}{A_2}\frac{dA_2}{dz_2}\right)G_{z_1} + \left(\frac{U_{z_1z_1z_1}}{2U_{z_1}} + 2U_{z_1} - 2\frac{A_3}{A_2^2}\right)G_{z_1} \\ & + \left(\frac{4\lambda A_3}{3A_2^2} - \frac{2\lambda}{3A_2}U_{z_1} - \frac{8\lambda^3}{27A_2^3} - \frac{1}{2A_2}\frac{dA_2}{dz_2}\right)G,\end{aligned} \tag{31}$$

whose compatibility condition yields the $1+1$ equation

$$\left(U_{z_2} - U_{z_1z_1z_1} + \frac{3}{4}\frac{U_{z_1z_1}^2}{U_{z_1}} - 3U_{z_1}^2 + \frac{2A_3}{A_2}U_{z_1} + \frac{(z_1U_{z_1} + U)}{A_2}\frac{dA_2}{dz_2}\right)_{z_1} = 0. \tag{32}$$

Setting $U_{z_1} = -P^2$ we have the equation:

$$P_{z_2} - P_{z_1z_1z_1} + 6P^2P_{z_1} + \frac{2A_3}{A_2}P_{z_1} + \frac{1}{A_2}\frac{dA_2}{dz_2}(P + z_1P_{z_1}) = 0.$$

The explicit dependence on $A_2$ and $A_3$ can be removed by doing the following change of variables:

$$T = \int \frac{dz_2}{A_2^3}, \qquad X = \frac{1}{A_2} z_1 - 2\int \frac{A_3}{A_2^2}\, dz_2, \qquad P(z_1, z_2) = \frac{Q(T, X)}{A_2},$$

that transforms the above equation in the modified Korteweg–de Vries equation

$$Q_T - Q_{XXX} + 6Q^2 Q_X = 0. \tag{33}$$

**Case 4.** $A_1 = 0$, $A_2 \neq 0$, $\beta = 0$.

The reduction is

$$z_1 = y, \qquad z_2 = z, \tag{34}$$

$$\psi(x, y, z) = e^{-\frac{2\lambda}{3A_2}x} G(z_1, z_2), \tag{35}$$

$$h(x, y, z) = U(z_1, z_2) - \frac{1}{6A_2}\frac{dA_2}{dy}x^2 - \frac{2}{3}\frac{A_3}{A_2}x, \tag{36}$$

whose substitution in (2) gives us

$$G_{z_1} = \left(\frac{4\lambda A_3}{3A_2^2} - \frac{8\lambda^3}{27A_2^3} - \frac{1}{A_2}\frac{dA_2}{dz_1}\right)G, \qquad G_{z_2} = \frac{3A_2}{2\lambda}U_{z_2}G. \tag{37}$$

The compatibility of (37) provides the linear equation

$$A_2 U_{z_1 z_2} + \frac{dA_2}{dz_1}U_{z_2} = 0 \Rightarrow U(z_1, z_2) = \frac{K_1(z_2)}{A_2} + F_1(z_1), \tag{38}$$

where $F_1(z_1)$ and $K_1(z_2)$ are arbitrary functions. With the aid of (38) we can integrate (37) as:

$$G = \alpha_0 \frac{1}{\sqrt{A_2}} \exp\left(\frac{3K_1}{2\lambda} + \frac{4\lambda}{3}\int \frac{A_3}{A_2^2}\, dz_1 - \frac{8\lambda^3}{27}\int \frac{1}{A_2^3}\, dz_1\right), \tag{39}$$

where $\alpha_0$ is an arbitrary constant. The resulting solution for (2) is

$$h = \frac{K_1}{A_2} + F_1 - \frac{1}{6A_2}\frac{dA_2}{dy}x^2 - \frac{2}{3}\frac{A_3}{A_2}x, \tag{40}$$

$$\psi = \frac{\alpha_0}{\sqrt{A_2}} \exp\left(\frac{3K_1}{2\lambda} + \frac{4\lambda}{3}\int \frac{A_3}{A_2^2}\, dy - \frac{8\lambda^3}{27}\int \frac{1}{A_2^3}\, dy - \frac{2\lambda}{3A_2}x\right), \tag{41}$$

depending on two arbitrary constants $\alpha_0$ and $\lambda$ and five arbitrary functions $A_1(y)$, $A_2(y)$, $A_3(y)$, $F_1(y)$, and $K_1(z)$.

**Case 5.** $A_1 = 0$, $A_2 = 0$, $\beta \neq 0$.

Integration of (7) provides the following reduction:

$$z_1 = x, \qquad z_2 = y, \tag{42}$$

$$\psi(x, y, z) = e^{\lambda \int \frac{dz}{\beta}} G(z_1, z_2), \qquad h(x, y, z) = U(z_1, z_2) + A_3 \int \frac{dz}{\beta}. \tag{43}$$

Substitution of the reduction ansatz (42), (43) in (2) gives us:

$$0 = G_{z_1 z_1 z_1} - G_{z_2} + 3U_{z_1}G_{z_1} + \frac{3}{2}U_{z_1 z_1}G, \qquad 0 = \lambda G_{z_1} + A_3 G. \tag{44}$$

The compatibility condition yields the linear equation

$$\lambda U_{z_1 z_1 z_1} - 2A_3 U_{z_1 z_1} + \frac{2}{3}\frac{dA_3}{dz_2} = 0 \tag{45}$$

that can be easily integrated as:

$$U(z_1, z_2) = F_3(z_2) e^{\frac{2A_3}{\lambda} z_1} + \frac{1}{6A_3}\frac{dA_3}{dz_2} z_1^2 + \left(\frac{\lambda}{6A_3^2}\frac{dA_3}{dz_2} + \frac{1}{2A_3}F_1(z_2)\right) z_1 + \frac{\lambda^2}{12A_3^3}\frac{dA_3}{dz_2} + \frac{\lambda}{4A_3^2}F_1(z_2) + \frac{1}{2A_3}F_2(z_2), \tag{46}$$

where $F_1(z_2)$, $F_2(z_2)$ and $F_3(z_2)$ are arbitrary functions. Expression (46) allows us to integrate (44) as:

$$G = \alpha_0 \exp\left(-\left(\frac{3}{2\lambda}\int F_1\, dz_2 + \frac{1}{\lambda^3}\int A_3^3\, dz_2 + \frac{A_3}{\lambda} z_1\right)\right), \tag{47}$$

where $\alpha_0$ is an arbitrary constant. We obtain the following solution for (2):

$$h = F_3 e^{\frac{2A_3}{\lambda} x} + \frac{1}{6A_3}\frac{dA_3}{dy} x^2 + \left(\frac{\lambda}{6A_3^2}\frac{dA_3}{dy} + \frac{1}{2A_3}F_1\right) x + \frac{\lambda^2}{12A_3^3}\frac{dA_3}{dy} + \frac{\lambda}{4A_3^2}F_1 + \frac{1}{2A_3}F_2 + A_3\int \frac{dz}{\beta}, \tag{48}$$

$$\psi = \alpha_0 \exp\left(\lambda \int \frac{dz}{\beta} - \left(\frac{3}{2\lambda}\int F_1\, dy + \frac{1}{\lambda^3}\int A_3^3\, dy + \frac{A_3}{\lambda} x\right)\right), \tag{49}$$

depending on two arbitrary constants $\alpha_0$ and $\lambda$ and six arbitrary functions $A_1(y)$, $A_2(y)$, $A_3(y)$, $F_1(y)$, $F_2(y)$, $F_3(y)$ and $\beta(z)$.

## 4. Conclusions

- A spectral problem in $2+1$ dimensions is presented. It should be noted that this Lax pair was obtained using the Singular Manifold Method that, surprisingly, does not work properly for some $1+1$ reductions of the system. It is easier to solve the problem in $2+1$ than in $1+1$ dimensions.
- For the above reason we attempt to go from $2+1$ to $1+1$ dimensions by using the reductions arising from the classical Lie symmetries of the Lax pair. This means that we obtain symmetries that are symmetries of both the field $h$ and the eigenfunction $\psi$. The symmetries that we have obtained include several arbitrary functions as well as an arbitrary constant that plays the role of the spectral parameter of the reduced spectral problems.
- Five possible reductions arise from the classical symmetries. Two of them yield linear equations that can be easily integrated, providing us with nontrivial solutions of (2).
- The other three reductions yield interesting Lax pairs of second, third and fourth order, respectively. An important point is that the introduction of the spectral parameters in the $1+1$ reductions arise in an absolutely natural way. The reduced Eqs. (16), (22) and (33) do not depend of $A_i(y)$, $\beta(z)$ which means that every different functional form of these functions gives raise to a different solution of the $2+1$ solution $h(x, y, z)$ by means of the corresponding reduction ansatze.

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