Theoretical and Mathematical Physics, **151**(3): 754–762 (2007)

ALGORITHMIC CONSTRUCTION OF LUMPS

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We use the singular manifold method to generate lump solutions of a Schrödinger equation in 2+1 dimensions and present three different types of such solutions.

Keywords: lump solution, singular manifold method, Schrödinger equation

1. Introduction

Our main objective in this paper is to construct an algorithmic procedure based in the singular manifold method (SMM) [1] and allowing rational solitons (lumps) to be found for equations in 2+1 dimensions. Lumps have been actively studied in the last few years (see, e.g., [2]).

We consider the 2+1 system [3], [4]

$$\begin{aligned} &m_y + uw = 0,\\ &iu_t + u_{xx} + 2um_x = 0,\\ &-iw_t + w_{xx} + 2wm_x = 0. \end{aligned} \tag{1}$$

Our interest in this system is motivated by the following considerations. If m is real and w is the complex conjugate of u, then (1) is the equation proposed by Fokas [4], which reduces to the nonlinear Schrödinger equation for $x = y$. Real and complex versions of (1) were respectively discussed in [5] and [6]. The Painlevé property was investigated in [7] and [8]. Line soliton solutions and dromions were obtained using the SMM [3] and the Hirota method [7]. Miura transformations between (1) and the generalized dispersive wave equation [9] were presented in [10]. Darboux transformations appeared in [3], where the SMM was deeply studied.

This paper is organized as follows. In Sec. 2, we present the SMM for (1). In Sec. 3, we use the SMM as a procedure to iterate fields, eigenfunctions, and singular manifolds such that new solutions arise iteratively from trivial seed solutions. Darboux transformations and Lax pairs also appear naturally. In Sec. 4, we generate lump solutions of (1) using this method.

2. Singular manifold method

We apply the SMM to (1) as was done in [10] but present a more convenient version of these results.

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Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 151, No. 3, pp. 387–395, June, 2007.

2.1. Truncated expansion. The SMM implies that the fields can be expanded as a truncated Painlevé series of the form

$$
\begin{aligned}
u^{(1)} &= u^{(0)} + A^{(0)} \frac{\phi_x^{(0)}}{\phi^{(0)}}, \\
w^{(1)} &= w^{(0)} + B^{(0)} \frac{\phi_x^{(0)}}{\phi^{(0)}}, \\
m^{(1)} &= m^{(0)} + \frac{\phi_x^{(0)}}{\phi^{(0)}},
\end{aligned} \tag{2}
$$

where $\{m^{(0)}, u^{(0)}, w^{(0)}\}$ is a seed solution and $\phi^{(0)}(x,y,t)$ is the singular manifold corresponding to this solution. The functions $A^{(0)}(x,y,t)$ and $B^{(0)}(x,y,t)$ are to be determined by substituting (2) in (1); $\{m^{(1)}, u^{(1)}, w^{(1)}\}$ is a new solution of (1) obtained via auto-Bäcklund transformation (2). The indices (0) and (1) correspond to the seed solution and the first iteration, the index (n) corresponds to n iterations of the auto-Bäcklund transformation (2), and $\phi^{(n)}$ therefore denotes the singular manifold corresponding to the solution $\{m^{(n)}, u^{(n)}, w^{(n)}\}$.

For any (n), we define the quantities

$$
v^{(n)} = \frac{\phi_{xx}^{(n)}}{\phi_x^{(n)}}, \qquad q^{(n)} = \frac{\phi_y^{(n)}}{\phi_x^{(n)}}, \qquad r^{(n)} = \frac{\phi_t^{(n)}}{\phi_x^{(n)}} \tag{3}
$$

and also the Schwarzian derivative

$$
s^{(n)} = v_x^{(n)} - \frac{\left(v^{(n)}\right)^2}{2}. \tag{4}
$$

We then have the relations

$$
\begin{aligned}
&r_y^{(n)} - q_t^{(n)} + r^{(n)} q_x^{(n)} - q^{(n)} r_x^{(n)} = 0, \\
&v_y^{(n)} = \left(q_x^{(n)} + q^{(n)} v^{(n)}\right)_x, \\
&v_y^{(n)} = \left(q_x^{(n)} + q^{(n)} v^{(n)}\right)_x.
\end{aligned} \tag{5}
$$

2.2. Seed solutions and singular manifold equations. Directly substituting (2) in (1) gives three polynomials in $\phi^{(0)}$. Setting each coefficient of these polynomials to zero and using (3)–(5), we obtain several equations. Using MAPLE to manipulate them, we obtain the singular manifold equations

$$
\begin{aligned}
q^{(0)} &= A^{(0)} B^{(0)}, \\
r_x^{(0)} &= \frac{1}{2}\left(\frac{i A_{xx}^{(0)}}{A^{(0)}} - i \frac{B_{xx}^{(0)}}{B^{(0)}} - \frac{A_t^{(0)}}{A^{(0)}} - \frac{B_t^{(0)}}{B^{(0)}} \right), \\
s^{(0)} &= -\frac{A_{xx}^{(0)}}{A^{(0)}} - \frac{B_{xx}^{(0)}}{B^{(0)}} - i \frac{A_t^{(0)}}{A^{(0)}} + i \frac{B_t^{(0)}}{B^{(0)}} - \frac{\left(r^{(0)}\right)^2}{2} + \int r_t^{(0)}\, dx
\end{aligned} \tag{6}
$$

and the expressions

$$
\begin{aligned}
u^{(0)} &= -\frac{A^{(0)}}{2}\left(i r^{(0)} + v^{(0)} + \frac{A_x^{(0)}}{A^{(0)}} \right), \\
w^{(0)} &= -\frac{B^{(0)}}{2}\left(-i r^{(0)} + v^{(0)} + \frac{B_x^{(0)}}{B^{(0)}} \right), \\
m_x^{(0)} &= -\frac{1}{4}\left(2 v_x^{(0)} + \frac{A_{xx}^{(0)}}{A^{(0)}} + \frac{B_{xx}^{(0)}}{B^{(0)}} + i \frac{A_t^{(0)}}{A^{(0)}} - i \frac{B_t^{(0)}}{B^{(0)}} \right),
\end{aligned}
$$

which can be linearized, yielding the Lax pair.

2.3. Lax pair. Introducing two functions $\psi^{(0)}$ and $\varphi^{(0)}$ defined by

$$v^{(0)} = \frac{\psi_x^{(0)}}{\psi^{(0)}} + \frac{\varphi_x^{(0)}}{\varphi^{(0)}}, \qquad r^{(0)} = i\biggl(\frac{\psi_x^{(0)}}{\psi^{(0)}} - \frac{\varphi_x^{(0)}}{\varphi^{(0)}}\biggr)$$

(see [3]) and proceeding as in [3], we obtain

$$\begin{aligned}
&u^{(0)}\psi_{xy}^{(0)} - u_x^{(0)}\psi_y^{(0)} - \bigl(u^{(0)}\bigr)^2 w^{(0)}\psi^{(0)} = 0,\\
&w^{(0)}\varphi_{xy}^{(0)} - w_x^{(0)}\varphi_y^{(0)} - \bigl(w^{(0)}\bigr)^2 u^{(0)}\varphi^{(0)} = 0,\\
&i\psi_t^{(0)} + \psi_{xx}^{(0)} + 2m_x^{(0)}\psi^{(0)} = 0,\\
&-i\varphi_t^{(0)} + \varphi_{xx}^{(0)} + 2m_x^{(0)}\varphi^{(0)} = 0.
\end{aligned} \tag{7}$$

It is trivial to verify that the compatibility condition for (7) is that $\{u^{(0)}, w^{(0)}, m^{(0)}\}$ satisfies (1). In fact, (7) is a two-component Lax pair for (1).

Computing $A^{(0)}$ and $B^{(0)}$ and substituting them in (2), we obtain

$$\begin{aligned}
&u^{(1)} = u^{(0)} - \frac{1}{w^{(0)}}\frac{\psi^{(0)}\varphi_y^{(0)}}{\phi^{(0)}},\\
&w^{(1)} = w^{(0)} - \frac{1}{u^{(0)}}\frac{\varphi^{(0)}\psi_y^{(0)}}{\phi^{(0)}},\\
&m^{(1)} = m^{(0)} + \frac{\phi_x^{(0)}}{\phi^{(0)}},
\end{aligned} \tag{8}$$

where $\psi^{(0)}$ and $\varphi^{(0)}$ are eigenfunctions of the seed solution $\{u^{(0)}, w^{(0)}, m^{(0)}\}$ and $\phi^{(0)}$ can be determined in terms of these eigenfunctions via the exact derivative

$$d\phi^{(0)} = \psi^{(0)}\varphi^{(0)}\,dx + \frac{1}{u^{(0)}w^{(0)}}\psi_y^{(0)}\varphi_y^{(0)}\,dy + i\bigl(\varphi^{(0)}\psi_x^{(0)} - \psi^{(0)}\varphi_x^{(0)}\bigr)\,dt. \tag{9}$$

2.4. Darboux transformations. We continue the procedure developed in [3]. Let $\psi_1^{(0)}, \varphi_1^{(0)}$ and $\psi_2^{(0)}, \varphi_2^{(0)}$ be two different pairs of eigenfunctions for the seed Lax pair. Equality (9) allows constructing two zeroth-order eigenfunctions $\phi_1^{(0)}$ and $\phi_2^{(0)}$ via the expression

$$d\phi_j^{(0)} = \psi_j^{(0)}\varphi_j^{(0)}\,dx + \frac{1}{u^{(0)}w^{(0)}}\bigl(\psi_j^{(0)}\bigr)_y\bigl(\varphi_j^{(0)}\bigr)_y\,dy + i\bigl[\varphi_j^{(0)}\bigl(\psi_j^{(0)}\bigr)_x - \psi_j^{(0)}\bigl(\varphi_j^{(0)}\bigr)_x\bigr]\,dt. \tag{10}$$

Using $\psi_1^{(0)}$ and $\varphi_1^{(0)}$ in (8), we obtain an iterated solution $\{u^{(1)}, w^{(1)}, m^{(1)}\}$ that suggests an iteration of the eigenfunctions of the form (see [3], [10], [11])

$$\psi_2^{(1)} = \psi_2^{(0)} - \psi_1^{(0)}\frac{\Omega_{1,2}^{(0)}}{\phi_1^{(0)}}, \qquad \varphi_2^{(1)} = \varphi_2^{(0)} - \varphi_1^{(0)}\frac{\Delta_{1,2}^{(0)}}{\phi_1^{(0)}} \tag{11}$$

such that $\psi_2^{(1)}$ and $\varphi_2^{(1)}$ are eigenfunctions for the iterated solution $\{u^{(1)}, w^{(1)}, m^{(1)}\}$ obtained by the Painlevé expansion of the seed eigenfunctions $\psi_2^{(0)}$ and $\varphi_2^{(0)}$. Furthermore, the singular manifold itself can be iterated as

$$\phi_2^{(1)} = \phi_2^{(0)} - \frac{\Omega_{1,2}^{(0)}\Delta_{1,2}^{(0)}}{\phi_1^{(0)}}$$

such that $\phi_2^{(1)}$ is the singular manifold that arises from $\psi_2^{(1)}$ and $\varphi_2^{(1)}$ via expression (10) with the index (0) replaced with (1) and $j=2$. In addition, we obtain the expressions for $\Omega_{i,j}^{(0)}$ and $\Delta_{i,j}^{(0)}$, $i,j=1,2$,

$$d\Omega_{i,j}^{(0)} = \psi_j^{(0)}\varphi_i^{(0)}\,dx + \frac{1}{u^{(0)}w^{(0)}}\bigl(\psi_j^{(0)}\bigr)_y\bigl(\varphi_i^{(0)}\bigr)_y\,dy + i\bigl[\varphi_i^{(0)}\bigl(\psi_j^{(0)}\bigr)_x - \psi_j^{(0)}\bigl(\varphi_i^{(0)}\bigr)_x\bigr]\,dt,$$

$$\Delta_{i,j}^{(0)} = \Omega_{j,i}^{(0)}.$$

We note that $\phi_j^{(0)} = \Omega_{j,j}^{(0)}$.

3. Iteration

3.1. Second iteration. Because $\phi_2^{(1)}$ is a singular manifold for the iterated solution $\{u^{(1)}, w^{(1)}, m^{(1)}\}$, we can iterate again and obtain the second-order iterated solution

$$u^{(2)} = u^{(0)} - \frac{1}{w^{(0)}}\bigl[\bigl(\varphi_1^{(0)}\bigr)_y\bigl(\psi_1^{(0)}\phi_2^{(0)} - \psi_2^{(0)}\Omega_{2,1}^{(0)}\bigr) + \bigl(\varphi_2^{(0)}\bigr)_y\bigl(\psi_2^{(0)}\phi_1^{(0)} - \psi_1^{(0)}\Omega_{1,2}^{(0)}\bigr)\bigr]\frac{1}{\tau_{1,2}},$$

$$w^{(2)} = w^{(0)} - \frac{1}{u^{(0)}}\bigl[\bigl(\psi_1^{(0)}\bigr)_y\bigl(\varphi_1^{(0)}\phi_2^{(0)} - \varphi_2^{(0)}\Omega_{1,2}^{(0)}\bigr) + \bigl(\psi_2^{(0)}\bigr)_y\bigl(\varphi_2^{(0)}\phi_1^{(0)} - \varphi_1^{(0)}\Omega_{2,1}^{(0)}\bigr)\bigr]\frac{1}{\tau_{1,2}},$$

$$m^{(2)} = m^{(0)} + \frac{(\tau_{1,2})_x}{\tau_{1,2}},$$

where

$$\tau_{1,2} = \phi_2^{(1)}\phi_1^{(0)} = \phi_1^{(0)}\phi_2^{(0)} - \Omega_{1,2}^{(0)}\Omega_{2,1}^{(0)} = \det\begin{pmatrix}\Omega_{1,1}^{(0)} & \Omega_{1,2}^{(0)}\\ \Omega_{2,1}^{(0)} & \Omega_{2,2}^{(0)}\end{pmatrix}.$$

The matrix element $\Omega_{i,j}^{(0)}$ can be also iterated, i.e.,

$$d\Omega_{i,j}^{(1)} = \psi_j^{(1)}\varphi_i^{(1)}\,dx + \frac{1}{u^{(1)}w^{(1)}}\bigl(\psi_j^{(1)}\bigr)_y\bigl(\varphi_i^{(1)}\bigr)_y\,dy + i\bigl[\varphi_i^{(1)}\bigl(\psi_j^{(1)}\bigr)_x - \psi_j^{(1)}\bigl(\varphi_i^{(1)}\bigr)_x\bigr]\,dt.$$

Using (11), we can easily verify that the truncated Painlevé expansion for the matrix element is

$$\Omega_{i,j}^{(1)} = \Omega_{i,j}^{(0)} - \frac{\Omega_{i,1}^{(0)}\Omega_{1,j}^{(0)}}{\phi_1^{(0)}}.$$

3.2. The $(n+1)$th iteration. The above procedure can be easily iterated n times. If $\psi_h^{(n)}$ and $\varphi_h^{(n)}$ are two eigenfunctions of the nth iteration and $\phi_h^{(n)}$ is the corresponding singular manifold, then we can summarize the results as

$$u^{(n+1)} = u^{(n)} - \frac{1}{w^{(n)}}\frac{\psi_h^{(n)}\bigl(\varphi_h^{(n)}\bigr)_y}{\phi_h^{(n)}}, \qquad w^{(n+1)} = w^{(n)} - \frac{1}{u^{(n)}}\frac{\varphi_h^{(n)}\bigl(\psi_h^{(n)}\bigr)_y}{\phi_h^{(n)}},$$

$$m^{(n+1)} = m^{(n)} + \frac{\bigl(\phi_h^{(n)}\bigr)_x}{\phi_h^{(n)}}, \qquad \psi_j^{(n+1)} = \psi_j^{(n)} - \psi_h^{(n)}\frac{\Omega_{h,j}^{(n)}}{\phi_h^{(n)}},$$

$$\varphi_j^{(n+1)} = \varphi_j^{(n)} - \varphi_h^{(n)}\frac{\Omega_{j,h}^{(n)}}{\phi_h^{(n)}}, \qquad \phi_j^{(n+1)} = \phi_j^{(n)} - \frac{\Omega_{j,h}^{(n)}\Omega_{h,j}^{(n)}}{\phi_h^{(n)}},$$

$$\Omega_{i,j}^{(n+1)} = \Omega_{i,j}^{(n)} - \frac{\Omega_{i,h}^{(n)}\Omega_{h,j}^{(n)}}{\phi_h^{(n)}},$$

where $\Omega_{h,j}^{(n)}$ is the matrix defined via the exact derivative,

$$d\Omega_{h,j}^{(n)} = \psi_j^{(n)}\varphi_h^{(n)}\,dx + \frac{1}{u^{(n)}w^{(n)}}\big(\psi_j^{(n)}\big)_y\big(\varphi_h^{(n)}\big)_y\,dy + i\big[\varphi_h^{(n)}\big(\psi_j^{(n)}\big)_x - \psi_j^{(n)}\big(\varphi_h^{(n)}\big)_x\big]\,dt,$$

and $\phi_j^{(n)} = \Omega_{j,j}^{(n)}$.

4. Lumps

In this section, we use the method to obtain lumps. Taking $u^{(0)} = 1$, $w^{(0)} = 1$, and $m^{(0)} = -y$ as the seed solution, we obtain

$$\begin{aligned}
&\big(\psi_j^{(0)}\big)_{xy} - \psi_j^{(0)} = 0,\\
&\big(\varphi_j^{(0)}\big)_{xy} - \varphi_j^{(0)} = 0,\\
&i\big(\psi_j^{(0)}\big)_t + \big(\psi_j^{(0)}\big)_{xx} = 0,\\
&-i\big(\varphi_j^{(0)}\big)_t + \big(\varphi_j^{(0)}\big)_{xx} = 0.
\end{aligned} \tag{12}$$

The eigenvalues for (12) are

$$\psi_j^{(0)} = e^{k_jQ(k_j)}[\alpha_j + \beta_jP(k_j)], \qquad \varphi_j^{(0)} = e^{-n_jQ(n_j)}[\gamma_j + \delta_jP(n_j)], \quad j = 1, 2,$$

where k_j, n_j, α_j, β_j, δ_j, and γ_j are complex constants and P and Q are the polynomials

$$P(k_j) = x - \frac{y}{k_j^2} + 2ik_jt, \qquad Q(k_j) = x + \frac{y}{k_j^2} + ik_jt.$$

Because we seek rational solutions (lumps), we need a polynomial expression for the singular manifolds, and it is therefore clear that $n_j = k_j$. We present three different cases.

4.1. Lumps of type I. Setting $\alpha_1 = \gamma_1 = 1$, $\beta_1 = \delta_1 = 0$, $\alpha_2 = \gamma_2 = 1$, $\beta_2 = \delta_2 = 0$, $k_1 = n_1$, and $k_2 = n_2 = -k_1^*$ (the asterisk denotes complex conjugation), we obtain

$$\begin{aligned}
&\psi_1^{(0)} = e^{k_1Q_1}, \qquad \varphi_1^{(0)} = e^{-k_1Q_1}, \qquad \psi_2^{(0)} = e^{-k_1^*Q_1^*}, \qquad \varphi_2^{(0)} = e^{k_1^*Q_1^*},\\
&Q_1 = x + \frac{y}{k_1^2} + ik_1t, \qquad P_1 = x - \frac{y}{k_1^2} + 2ik_1t.
\end{aligned}$$

Therefore, we can compute the matrix $\Omega_{i,j}^{(0)}$:

$$\begin{aligned}
d\phi_1^{(0)} &= d\Omega_{1,1}^{(0)} = dx - \frac{1}{k_1^2}\,dy + 2ik_1\,dt,\\
d\phi_2^{(0)} &= d\Omega_{2,2}^{(0)} = dx - \frac{1}{(k_1^*)^2}\,dy + 2ik_1^*\,dt,\\
d\Omega_{1,2}^{(0)} &= \left[dx + \frac{1}{k_1k_1^*}\,dy + i(k_1 - k_1^*)\,dt\right]e^{-k_1Q_1}e^{-k_1^*Q_1^*},\\
d\Omega_{2,1}^{(0)} &= \left[dx + \frac{1}{k_1k_1^*}\,dy + i(k_1 - k_1^*)\,dt\right]e^{k_1Q_1}e^{k_1^*Q_1^*}.
\end{aligned}$$

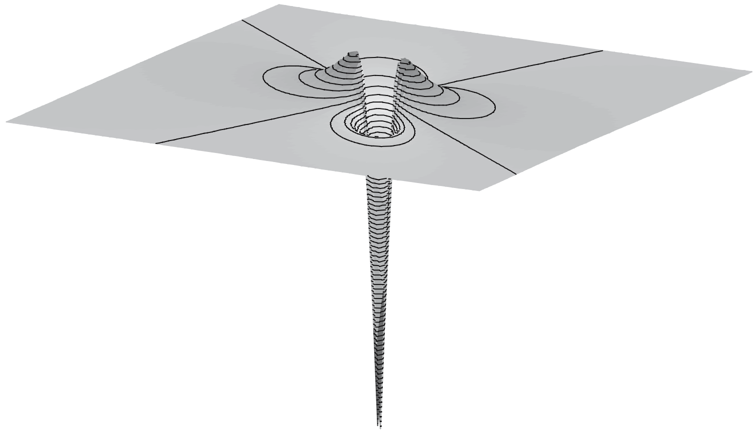

Fig. 1. Lumps of type I.

It can be integrated as

$$\Omega_{1,1}^{(0)} = \phi_1^{(0)} = P_1, \qquad \Omega_{2,2}^{(0)} = \phi_2^{(0)} = P_1^*,$$

$$\Omega_{1,2}^{(0)} = -\frac{1}{k_1 + k_1^*} e^{-k_1 Q_1} e^{-k_1^* Q_1^*}, \qquad \Omega_{2,1}^{(0)} = \frac{1}{k_1 + k_1^*} e^{k_1 Q_1} e^{k_1^* Q_1^*}.$$

Hence, the function $\tau_{1,2}$ is

$$\tau_{1,2} = P_1 P_1^* + \left(\frac{1}{k_1 + k_1^*} \right)^2,$$

which can be written as the real positive-definite expression

$$\tau_{1,2} = X_1^2 + Y_1^2 + \left(\frac{1}{2a_1} \right)^2,$$

by setting $P_1 = X_1 + iY_1$, where $k_1 = a_1 + ib_1$ and

$$X_1 = x - \frac{a_1^2 - b_1^2}{(a_1^2 + b_1^2)^2} y - 2b_1 t, \qquad Y_1 = \frac{2a_1 b_1}{(a_1^2 + b_1^2)^2} y + 2a_1 t.$$

Then the solution for the second iteration is

$$u^{(2)} = 1 - \frac{1}{\tau_{1,2}} \frac{1 + 2i(b_1 X_1 + a_1 Y_1)}{a_1^2 + b_1^2},$$

$$w^{(2)} = 1 - \frac{1}{\tau_{1,2}} \frac{1 - 2i(b_1 X_1 + a_1 Y_1)}{a_1^2 + b_1^2},$$

$$m^{(2)} = -y + \frac{(\tau_{1,2})_x}{\tau_{1,2}}.$$

An example of type-I lumps is shown in Fig. 1.

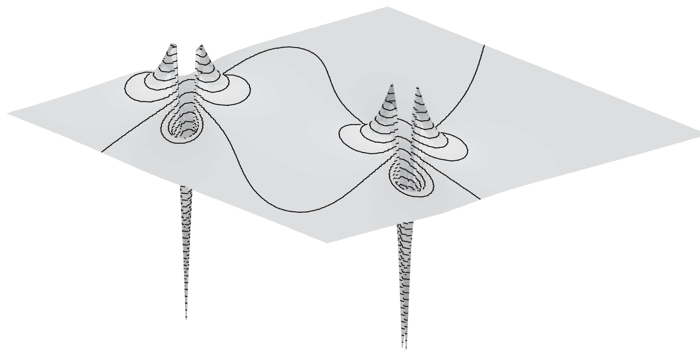

Fig. 2. Lumps of type II.

4.2. Lumps of type II. Setting $\alpha_1 = \gamma_1 = 1$, $\beta_1 = \delta_1 = 0$, $\alpha_2 = \gamma_2 = 0$, $\beta_2 = \delta_2 = 1$, $k_1 = n_1$, and $k_2 = n_2 = -k_1^*$, we obtain

$$\psi_1^{(0)} = e^{k_1 Q_1}, \qquad \varphi_1^{(0)} = e^{-k_1 Q_1} P_1, \qquad \psi_2^{(0)} = e^{-k_1^* Q_1^*} P_1^*, \qquad \varphi_2^{(0)} = e^{k_1^* Q_1^*},$$

$$\Omega_{1,1}^{(0)} = \frac{1}{2}\left[X_1^2 - Y_1^2 - \frac{a_1^2 - 3b_1^2}{a_1^2 + b_1^2}\,\frac{Y_1 - 2a_1 t}{b_1}\right] + i\left[X_1 Y_1 - t + \frac{3a_1^2 - b_1^2}{a_1^2 + b_1^2}\,\frac{Y_1 - 2a_1 t}{2a_1}\right],$$

$$\Omega_{2,2}^{(0)} = \phi_2^{(0)} = \left(\phi_1^{(0)}\right)^*,$$

$$\Omega_{1,2}^{(0)} = -\frac{1}{2a_1}\left[\left(X_1 + \frac{1}{2a_1}\right)^2 + Y_1^2 + \frac{1}{4a_1^2}\right]\frac{1}{e^{k_1 Q_1} e^{k_1^* Q_1^*}},$$

$$\Omega_{2,1}^{(0)} = \frac{1}{2a_1} e^{k_1 Q_1} e^{k_1^* Q_1^*},$$

$$\tau_{1,2} = \frac{1}{4}\left[X_1^2 - Y_1^2 - \frac{a_1^2 - 3b_1^2}{a_1^2 + b_1^2}\,\frac{Y_1 - 2a_1 t}{b_1}\right]^2 + \left[X_1 Y_1 - t + \frac{3a_1^2 - b_1^2}{a_1^2 + b_1^2}\,\frac{Y_1 - 2a_1 t}{2a_1}\right]^2 + \\ + \frac{1}{4a_1^2}\left[\left(X_1 + \frac{1}{2a_1}\right)^2 + Y_1^2 + \frac{1}{4a_1^2}\right].$$

An example of type-II lumps is shown in Fig. 2.

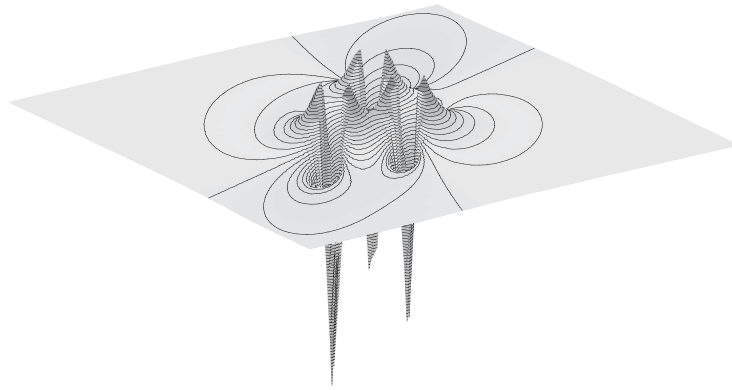

Fig. 3. Lumps of type III.

4.3. Lumps of type III. Setting $\alpha_1 = \delta_1 = 0$, $\beta_1 = \gamma_1 = 1$, $\alpha_2 = \delta_2 = 0$, $\beta_2 = \gamma_2 = 1$, $k_1 = n_1$, and $k_2 = n_2 = -k_1^*$, we obtain

$$\psi_1^{(0)} = e^{k_1 Q_1} P_1, \qquad \varphi_1^{(0)} = e^{-k_1 Q_1} P_1, \qquad \psi_2^{(0)} = e^{-k_1^* Q_1^*} P_1^*, \qquad \varphi_2^{(0)} = e^{k_1^* Q_1^*} P_1^*,$$

$$\Omega_{1,1}^{(0)} = \left[\frac{X_1^3}{3} - X_1 Y_1^2 + \frac{a_1^4 - 6a_1^2 b_1^2 + b_1^4}{2a_1 b_1 (a_1^2 + b_1^2)^2}(Y_1 - 2a_1 t)\right] + \\ + i\left[-\frac{Y_1^3}{3} + X_1^2 Y_1 - \frac{2(a_1^2 - b_1^2)}{(a_1^2 + b_1^2)^2}(Y_1 - 2a_1 t)\right],$$

$$\Omega_{2,2}^{(0)} = \phi_2^{(0)} = \left(\phi_1^{(0)}\right)^*,$$

$$\Omega_{1,2}^{(0)} = -\frac{1}{2a_1}\left[\left(X_1 + \frac{1}{2a_1}\right)^2 + Y_1^2 + \frac{1}{4a_1^2}\right] e^{-k_1 Q_1} e^{-k_1^* Q_1^*},$$

$$\Omega_{2,1}^{(0)} = \frac{1}{2a_1}\left[\left(X_1 - \frac{1}{2a_1}\right)^2 + Y_1^2 + \frac{1}{4a_1^2}\right] e^{k_1 Q_1} e^{k_1^* Q_1^*},$$

$$\tau_{1,2} = \left[\frac{X_1^3}{3} - X_1 Y_1^2 + \frac{a_1^4 - 6a_1^2 b_1^2 + b_1^4}{2a_1 b_1 (a_1^2 + b_1^2)^2}(Y_1 - 2a_1 t)\right]^2 + \\ + \left[-\frac{Y_1^3}{3} + X_1^2 Y_1 - \frac{2(a_1^2 - b_1^2)}{(a_1^2 + b_1^2)^2}(Y_1 - 2a_1 t)\right]^2 + \frac{1}{4a_1^2}\left[(X_1 + Y_1^2)^2 + \frac{Y_1^2}{a_1^2} + \frac{1}{4a_1^4}\right].$$

An example of type-III lumps is shown in Fig. 3.

In a previous paper [3], we saw that the one-soliton solution corresponds to the first iteration of a seed solution and the two-soliton solution aries from the second iteration such that each iteration provides a different wave number. Nevertheless, as we see here, obtaining a one-lump solution requires two iterations such that the second wave number is the complex conjugate of the first.

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