The Stochastic Burger's Equation in Ito's Sense

By Javier Villarroel

We consider the solution to Burger's equation coupled to a stochastic noise in Ito's sense. The main random properties of the wave are determined. The solution is related to a deterministic problem with a rescaled diffusion coefficient. Depending on the value of a parameter, the initial value problem may be ill posed, well posed up to an explosion time, or well posed for all time. Traveling waves are destroyed asymptotically by white noise. However, the only effect of colored noise is to render the wave position random.

1. Introduction

Because the seminal paper by Gardner et al. [1] (see also [2]) opened the possibility of integrating the well-known Korteweg–deVries (KdV) equation, the knowledge of "integrable" equations and the physical properties of their solutions has largely increased, see [3–5]. In [6] the relevant extension of the method to the multidimensional case was developed and used to find the solution of the initial value problem (i.v.p.) of the KPI equation. Foremost among the explicit solutions to these equations are the solitons; these are localized configurations that conserve shape upon time evolution. The underlying dynamics, characterized by the absence of asymptotic interaction, has been considered for long time the prototypical behavior for localized solutions of integrable equations. Recently, new localized solutions with

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nontrivial dynamics have been found for KPI and other integrable equations [7, 8]. We note that these solutions regain shape upon interaction, because it happens with standard solitons.

From a physical perspective a natural question is to determine if localized solutions are destroyed by the introduction of a stochastic noise $\zeta(t)$. In earlier work [9] it was shown that the KdV equation coupled to the noise $\zeta(t) \equiv W(t)$ in the derivative is still integrable, and that, in the sense of statistical average, the noise destroys the soliton as $t \to \infty$. Here $W(t)$ stands for the standard Brownian motion (B.M.) process, i.e., a stochastic process with a Gaussian distribution centered around the origin and with variance $\sigma^2(t) = t$.

In this paper we consider the general solution to another linearizable equation, namely, the Burgers equation coupled to a stochastic noise $\zeta(t)$ in the derivative

$$\frac{\partial u}{\partial t} + (u - \zeta)\frac{\partial u}{\partial z} - \frac{\alpha^2}{2}\frac{\partial^2 u}{\partial z^2} = 0 \tag{1}$$

with initial condition

$$u(t = 0, z) = \psi(z), \tag{2}$$

where $\psi(z)$ is a given function and α is a given constant.

Physical white noise corresponds to the formal choice $\zeta(t) = \dot{W}(t)$. However, it is well known that W is nowhere differentiable whence it follows that, as it stands, (1) is meaningless in this case. To overcome this ominous situation, a consistent theory of stochastic differential equations (SDE) must give a prescription to deal with this difficulty. Both from a mathematical and physical point of view the natural choices are the Stratonovitch theory, which obeys the rules of classical calculus, and Ito's theory, which does not. The latter interpretation is considered in this paper.

Equation (1) with the same kind of coupling has been considered in the paper [10] and interesting statistical averages have been computed. In this work it appears that, tacitly, the Stratonovitch convention was used.

Corresponding to $\zeta(t) = W(t)$, (1) has been studied in [11, 12] and the asymptotic behavior of the statistical average of some solutions has been determined. Because the sample paths, or trajectories, of $W(t)$ are continuous, it follows that in this case the solution $u(t, z)$ can be obtained using the rules of standard calculus irrespective of which interpretation is taken.

The analysis of the sample paths properties of $u(t, z)$ is beyond the study developed in [9–12] and has not so far been considered in detail. In particular, smoothness properties and the relevant densities of the amplitude, position and "passage times" of the front wave, remain to be determined.

In this paper we show that the solution to the Equation (1) coupled to nonsmooth noise in the Ito's context is reduced to that of a deterministic equation with a modified diffusion coefficient (see Equation (4) below). Thus, the effect of noise is to add a negative viscosity term, which renders the equation less diffusive. This is quite interesting and remains to be explained from a physical perspective. It would be interesting to determine if a connection with stochastic stability could be drawn up. For certain types of noise we can solve the resulting equation with total generality. It is found that any particular realization is a (random) translation of the unperturbed solution (see Equation (3)). The solution is continuous but nonsmooth and the time derivative does not exist in the usual sense. The main properties of the solution are determined; in particular, the probability density and the mean of the solution. Although our results apply to general types of noise, for the sake of being specific we are mostly concerned with white noise.

We find that if $\alpha = 1$, a shock may develop at a (deterministic) time and at a random point. If $\alpha < 1$ the corresponding i.v.p. is found to be ill posed. If $\alpha > 1$ the i.v.p. is well posed for all time. Particularly interesting is the case of the kink or ladder solution. We find that for long times the average effect of white noise is to flatten the solution through a region with width proportional to $\sqrt{t}$. For a general class of colored noises the situation is different. As $t \to \infty$ the random wave may either tend to the unperturbed solution or have a Gaussian location. This is very interesting; it proves that noise does not necessarily destroy localized solutions.

Results pertaining to the KP equation coupled to white noise in the derivative are also found to be equally interesting and will be presented elsewhere.

2. White noise: the method of solution

Consider the i.v.p. for Burger's equation coupled to white noise within the framework of Ito's theory:

$$\frac{\partial u}{\partial t} + (u - \dot{W}(t))\frac{\partial u}{\partial z} - \frac{\alpha^2}{2}\frac{\partial^2 u}{\partial z^2} = 0; \qquad u(t = 0, z) = \psi(z), \tag{1'}$$

where α is a constant.

The solution is given by

$$u(t, z) = g(t, z + W(t)), \tag{3}$$

where the deterministic function $g(t, z)$ solves

$$\frac{\partial g}{\partial t} + g\frac{\partial g}{\partial z} - \frac{\alpha^2 - 1}{2}\frac{\partial^2 g}{\partial z^2} = 0; \qquad g(t = 0, z) = \psi(z). \tag{4}$$

Remark 1: The above representation shows that the solution is continuous but nowhere differentiable.

Remark 2: If $\alpha > 1$ the use of Ito's theory versus Stratonovitch's amounts to the scaling of parameters $\alpha^2 \to \alpha^2 - 1 > 0$; hence our results are also relevant under the Stratonovitch interpretation. Note that some statistical averages of the solution to (1′) in the latter context have been considered in [10] (see also [13] and [14] for other developments).

Remark 3: When the noise is not white, a different equation for g is obtained. We briefly describe this case in Section 5.

Remark 4: To stress the random nature of the solution the notation $u(t, z, \omega)$ is sometimes preferred. Here $\omega \in \Omega$, where Ω is the space of elementary events on which a probability P is defined. (Actually P is a measure defined on a $\sigma-$ field $\mathcal{G}$ of subsets of Ω). In order not to overload the paper we skip a rigorous mathematical formalism; we refer the interested reader to the monographs [15–17].)

To prove (3) let $X(t) \equiv z + W(t)$ where we consider z as a parameter. Then $X(t)$ is one-dimensional B.M. starting from z. Let $g(t, z)$ solve (4) and define $Y(t) = g\ (t, X(t))$. Note that $Y(t)$ satisfies

$$\frac{\partial^n Y}{\partial z^n} = \frac{\partial^n g(t,x)}{\partial x^n}]_{x=X(t)}.$$

Using Ito's rule, we find that

$$\begin{aligned}\frac{\partial Y}{\partial t} &= \left[\frac{\partial g(t,x)}{\partial t} + \frac{1}{2}\frac{\partial^2 g(t,x)}{\partial x^2}\right]_{x=X(t)} + \dot{W}\frac{\partial g}{\partial x}\Big]_{x=X(t)} \\ &= \left[-g\frac{\partial g}{\partial x}(t,x) + \frac{\alpha^2}{2}\frac{\partial^2 g}{\partial x^2} + \dot{W}\frac{\partial g}{\partial x}\right]_{x=X(t)} = -Y\frac{\partial Y}{\partial z} + \frac{\alpha^2}{2}\frac{\partial^2 Y}{\partial z^2} + \dot{W}\frac{\partial Y}{\partial z}.\end{aligned}$$

3. Burger's equation with $\alpha = 1$

We consider first the case corresponding to $\alpha = 1$ in Equation ($1'$). The solution is given by (3) where the deterministic function $g(t, z)$ solves the classical shock-wave equation:

$$\frac{\partial g}{\partial t} + g\frac{\partial g}{\partial z}(t,z) = 0; \qquad g(t=0,z) = \psi(z). \tag{5}$$

It follows that

$$g(t,z) = \psi(z - gt), \tag{6}$$

which implies that $g(t, z)$ remains constant along characteristic lines: $g(t, a + \psi(a)t) = \psi(a)$. Points with amplitude $\psi(a)$ travel also with speed $\psi(a)$. Higher-points in the wave may eventually catch up and overtake those that have smaller amplitude. Thus the solution has to steepen up to time t_e at which the slope is vertical. In such a case the wave breaks and a shock is developed. The geometrical loci of the shock is the curve with parametric equations

$$t = -\frac{1}{\psi'(a)}; \qquad z = a - \frac{\psi(a)}{\psi'(a)}. \tag{7}$$

The smallest such t (if any) is termed explosion time t_e. Hence

$$t_e \equiv \begin{cases} \frac{-1}{\inf_a \psi'(a)} & \text{if } \inf_a \psi'(a) < 0 \\ \infty & \text{if } \inf_a \psi'(a) \geq 0. \end{cases} \tag{8}$$

We next determine the main statistical properties of $u(t, z)$. The most important of these is the probability density of $u(t, z)$ defined by $P\{u(t, z) \in dx\} = f^u(x)\,dx$. Remarkably, it can be evaluated in an explicit way for all initial data $\psi(z)$. Indeed, assume that for given x there exist $m(x)$ points $\rho_1(x), \ldots, \rho_m(x)$ such that $\psi(\rho_1(x)) = \cdots = \psi(\rho_m(x)) = x$. If $x \equiv g(t, z)$, (6) yields that there exist $m(x)$ points $h_j(t, x) = tx + \rho_j(x)$ such that $g(t, h_j(t, x)) = x$. Transforming the corresponding probability integral one has that

$$f^u(x) = \frac{1}{[2\pi t]^{1/2}} \sum_{j=1}^{m(x)} |t + \rho_j'(x)| \exp\left[-\frac{(tx + \rho_j(x) - z)^2}{2t}\right]. \tag{9}$$

The mean value $\mu(t, z)$ of the random wave $u(t, z)$ is given by

$$\mu(t, z) \equiv \langle u(t, z)\rangle = \frac{1}{[2\pi t]^{1/2}} \int dx\, g(t, x) \exp\left[-\frac{(x - z)^2}{2t}\right] \tag{10}$$

$$= \frac{1}{[2\pi t]^{1/2}} \int \sum_{j=1}^{m(x)} |t + \rho_j'(x)| x \exp\left[-\frac{(tx + \rho_j(x) - z)^2}{2t}\right] dx. \tag{11}$$

We note that $\langle u(t, z)\rangle$ satisfies the Fokker–Planck equation

$$\left[\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial z^2}\right]\langle u(t, z)\rangle = \left\langle \tfrac{\partial}{\partial t} u(t, z)\right\rangle$$
$$\lim_{t \downarrow 0}\langle u(t, z)\rangle = \psi(z).$$

Similarly one can compute the average of any function of the wave by

$$\langle F(u(t, z))\rangle = \frac{1}{[2\pi t]^{1/2}} \int dx\, F(g(t, x)) \exp\left[-\frac{(x - z)^2}{2t}\right]. \tag{12}$$

Finally if t_1, t_2 are two times the correlation function (in time) defined as

$$K(t_1, t_2) \equiv \langle u(t_1, z)u(t_2, z)\rangle - \langle u(t_1, z)\rangle\langle u(t_2, z)\rangle$$

can be shown to be given by

$$K(t_1, t_2) = \frac{1}{2\pi[(t_2 - t_1)t_1]^{1/2}} \int dx_1\, dx_2\, g(t_1, x_1)g(t_2, x_2)$$
$$\times \exp\left[-\frac{(x_2 - x_1 - z)^2}{2(t_2 - t_1)} - \frac{(x_1 - z)^2}{2t_1}\right] - \mu(t_1, z)\mu(t_2, z). \tag{13}$$

To be specific we consider the following example. Suppose that $\psi(z) \equiv u(0, z) = z$. Using (3) and (6), it follows that

$$u(t, z) = \frac{z + W(t)}{1 + t}. \tag{14}$$

If B is any set on R, (9) yields that

$$P[u(t, z) \in B] = \frac{1 + t}{[2\pi t]^{1/2}} \int_B \exp\left[-\frac{((t + 1)x - z)^2}{2t}\right] dx. \tag{15}$$

The mean value of the wave's amplitude is then given by

$$\langle u(t, z)\rangle = \frac{1}{[2\pi t]^{1/2}} \int \frac{x}{1 + t} \exp\left[-\frac{(x - z)^2}{2t}\right] dx = \frac{z}{1 + t} \equiv g(t, z). \tag{16}$$

Finally the variance of the solution and the time correlation function are found as

$$\sigma_u(t, z) \equiv \langle u^2(t, z)\rangle - \langle u(t, z)\rangle^2 = \frac{t}{(t + 1)^2}, \quad K(t, t_2) = \frac{\min(t_2, t_1)}{(t_1 + 1)(t_2 + 1)}. \tag{17}$$

In this example the solution will never break. Actually the average value “flattens” with time. As t goes to infinity all moments of the wave vanish.

Although statistical averages provide interesting information on the properties of the solution, questions of an analytical nature concerning typical realizations of the wave require a different study. We now discuss some of these properties. Notice that the solution remains constant along characteristics:

$$u(t, a - W(t) + \psi(a)t) = u(0, a). \tag{19}$$

The instant of explosion t_e and the explosion point z_e can be obtained by determining the envelope to the family of these characteristic curves. One finds that t_e is given by (8) and

$$z_e = l(t_e) - W(t_e) - \psi(l(t_e))t_e,$$

where

$$l(t) \equiv \psi'^{-1}\left(-\frac{1}{t}\right). \tag{20}$$

This is interesting. If the initial data satisfy $\inf_a \psi'(a) < 0$ the solution to (1′) will develop a shock due to the effect of noise; however the explosion time is deterministic and is not determined by the noise. Only the breaking point (20) is random with density

$$P(z_e \in dx) = \frac{1}{[2\pi t_e]^{1/2}} \exp\left[-\frac{(x - l(t_e) + \psi(l(t_e))t_e)^2}{2t_e}\right] dx. \tag{21}$$

Particularly interesting is the case corresponding to localized initial data. Assume that $\psi(z)$ is strongly localized around the point z_0. For sufficiently short times the solution remains localized, and the crest of the wave will be located at the random point

$$\tilde{z}(t) = z_0 - W(t) + \psi(z_0)t. \tag{22}$$

Thus, the wave experiences a Brownian motion around the unperturbed position $z_0 + \psi(z_0)\,t$. The time τ_b at which the wave crest first reaches a given point b satisfies

$$P[\tau_b \in dt] = [2\pi t^3]^{-1/2}|b - z_0| \exp\left[-\frac{(b - z_0 - \psi(z_0)t)^2}{2t}\right] dt \equiv f_\tau(t)\,dt. \tag{23}$$

By integration we obtain the probability that the crest of the wave ever reaches the point b

$$P(\tau_b < \infty) = \int_0^\infty f_\tau(t)\,dt = \begin{cases} e^{-2|b-z_0|\psi(z_0)}, & \text{if } b < z_0, \\ 1, & \text{if } b > z_0. \end{cases} \tag{24}$$

This means that only those points to the right of z_0 will be reached with probability 1, while those located to the left of z_0 are attained with a probability that decreases exponentially with the distance.

A related question is to determine the probability that at a given time t the wave has already arrived to a point $b \geq z_0$. Note that this happens iff $M \geq b$ where we define the statistical quantity $M \equiv \max_{t' \leq t} \tilde{z}(t')$. It can be proven that this probability equals

$$\frac{2}{[2\pi t]^{1/2}} \int_b^\infty dz \left[\exp\left[-\frac{(z - z_0 - \psi(z_0)t)^2}{2t}\right] \right. \\ \left. + 2\psi(z_0) \exp[-2\psi(z_0)(z - z_0)] \Phi\left(\frac{z_0 - z - \psi(z_0)t}{\sqrt{t}}\right)\right], \tag{25}$$

where Φ stands for the error function

$$\Phi(z) \equiv \frac{1}{[2\pi]^{1/2}} \int_{-\infty}^{z} \exp\left[-\frac{x^2}{2}\right] dx.$$

All these probabilities are obtained solving a partial differential equation with certain boundary conditions. Details are left for the Appendix.

As a specific example suppose that $\psi(z) = 1/(1 + z^2)$. In this case (6) amounts to the inversion of a cubic algebraic equation. Nevertheless we can determine the most important features of the motion. Using (9) with $m(x) = 2$, $x \in (0, 1)$ we obtain that if $0 \le \alpha \le \beta \le 1$,

$$P[\alpha \le u(t,z) \le \beta] = \frac{1}{[2\pi t]^{1/2}} \sum_{\pm} \int_\alpha^\beta e^{-\frac{1}{2t}\left(tx - z \pm \sqrt{\frac{1}{x-1}}\right)^2} \left| t \mp \frac{1}{2x^{\frac{3}{2}}\sqrt{1-x}} \right| dx \tag{26}$$

and that

$$\langle u(t,z)\rangle = \frac{1}{[2\pi t]^{1/2}} \sum_{\pm} \int_0^1 x e^{-\frac{1}{2t}\left(tx - z \pm \sqrt{\frac{1}{x}-1}\right)^2} \left| t \mp \frac{1}{2x^{\frac{3}{2}}\sqrt{1-x}} \right| dx. \tag{27}$$

A shock occurs at the time $t_e = \frac{8}{9}\sqrt{3}$ and develops from the random point $z_e = \sqrt{3} - W(\frac{8}{9}\sqrt{3})$. Before the shock regime sets in, i.e., for $t \ll t_e$, the random wave is localized around $\tilde{z}(t) = t - W(t)$. Thus it moves as B.M. with drift coefficient 1 starting from the origin.

4. Burgers equation: general case

We consider next the i.v.p. (1′) with $\alpha \neq 1$. The solution is given by (3) and (4). If $\alpha > 1$ the Cauchy problem (4) is well posed, while it is ill posed for $\alpha < 1$. In the sequel we assume that $\alpha > 1$. It is well known that $g(t, z)$ is given by

$$g(t,z) = (1-\alpha^2)\frac{\partial}{\partial z} \log \int_{-\infty}^{z} \exp\left[-\frac{(z-y)^2}{4\nu t} - \frac{1}{2\nu}\int_{-\infty}^{y} \psi(y')\,dy'\right] dy, \tag{28}$$

where $\nu \equiv \frac{\alpha^2-1}{2}$. Thus, the effect of considering the noise within the Ito's interpretation is to shift the diffusion parameter: $\alpha^2 \to \alpha^2 - 1 > 0$, and hence it results in the addition of a negative viscosity term which renders the equation less diffusive. Hence, upon this transformation, our results are also relevant under the Stratonovitch interpretation.

Assuming for convenience that $g(t, z)$ is an increasing function of z for every t, (3) implies that $u(t, z)$ has density

$$P\{u(t,z) \in dx\} = \frac{1}{[2\pi t]^{1/2}} \frac{\partial h}{\partial x} \exp\left[-\frac{(h(t,x)-z)^2}{2t}\right] dx, \tag{29}$$

where $g\ (t,\ h(t,\ x)) = x$. However, this expression is not complete until for given initial data the evaluation of the quadrature (28) and the inversion of the resulting function are performed; in contrast, for $\alpha = 1$, no further steps were necessary.

The long time limit of the mean of the solution is interesting. With enough decay on the initial data one has that

$$\lim_{t\to\infty} \langle u(t,z)\rangle = \frac{1}{[2\pi t]^{1/2}} \int dx \lim_{t\to\infty} g(t,x) \exp\left[-\frac{(x-z)^2}{2t}\right] = 0 \tag{30}$$

and also $\langle u(t,\ z)\rangle = o(1/t^n)$ whenever $\lim_{t\to\infty} g(t, x\sqrt{t})t^n = 0$ uniformly.

We consider next the behavior of the traveling wave solution of Burger's equation, often referred to as the "kink":

$$g(t,z) = v - 2\mu \tanh\left[\frac{\mu}{v}(z - vt)\right] \equiv g(z - vt). \tag{31}$$

Here v, μ are two constants that we assume to be positive. The derivative of this solution is a soliton localized along $z = vt$ which also constitutes the front wave of (30). The corresponding random wave is given by $g(z + W(t) - vt)$. The front wave position $\tilde{z}(t) = vt - W(t)$ is B.M. with drift parameter v. It follows that $\tilde{z}(t)$ and τ_b, the first arrival time to the point b, have density

$$f_{\tilde{z}(t)}(x) = \frac{1}{[2\pi t]^{1/2}} \exp\left[-\frac{(x - vt)^2}{2t}\right];\ f_\tau(t) = \frac{|b|}{[2\pi t^3]^{1/2}} \exp\left[-\frac{(b - vt)^2}{2t}\right]. \tag{32}$$

By integration we obtain the probability that the wave ever reaches the point b

$$\int_0^\infty f_\tau(t)\,dt = \begin{cases} e^{-2v|b|}, & \text{if } b < 0, \\ 1, & \text{if } b > 0 \end{cases} \tag{33}$$

and the corresponding mean values

$$\langle \tilde{z}(t) \rangle = vt; \quad \langle \tau_b \rangle = \int_0^\infty t f_\tau(t)\,dt = \begin{cases} \infty, & \text{if } b < 0, \\ \frac{b}{v}, & \text{if } b > 0. \end{cases} \tag{34}$$

The average effects of white noise for long times have been discussed in [10] and [12]. Note that in the frame $z = vt$ moving with the unperturbed wave the amplitude remains constant $g(t, z) = v$, and so does the time average: $\langle u(t, z) \rangle = g(t, z)$ (see [12]). However, one should not be misled into thinking that the noise has no significant effect on the wave; to substantiate this, note that on the region

$$z = vt + x_0\sqrt{t}, \tag{35}$$

where x_0 is arbitrary, is

$$\begin{aligned} &\lim_{t\to\infty} \langle u(t,z) \rangle - g(t,z) \\ &\quad = \int_{-\infty}^{-x_0} + \int_{-x_0}^{\infty} \frac{dx}{[2\pi]^{1/2}} \lim_{t\to\infty} (g((x + x_0)\sqrt{t}) - g(t,z)) e^{-\frac{x^2}{2}} \\ &\quad = \begin{cases} 4\mu[\Phi(-x_0) - 1], & x_0 < 0, \\ 4\mu\Phi(-x_0), & x_0 > 0, \\ 0, & x_0 = 0, \end{cases} \end{aligned} \tag{36}$$

which is in general different from zero. Likewise we find that $\lim_{t\to\infty}\langle u(t, z) - v\rangle^{2n+1} = -2\mu[1-2\Phi(-x_0)]$, $\lim_{t\to\infty}\langle u(t, z)-v\rangle^{2n} = -2\mu$. The interpretation of this result is the following. For long times the kink is still centered at the unperturbed position, but due to the effect of noise it flattens over a region with a width of order $O(\sqrt{t})$. Thus the effect of white noise is to destroy the kink. This interpretation is confirmed considering the behavior of the soliton derivative:

$$-\frac{\partial u}{\partial z}(t,z) = 2\frac{\mu^2}{\nu}\mathrm{sech}^2\left[\frac{\mu}{\nu}(z + W(t) - vt)\right]. \tag{37}$$

If z is kept constant it is possible to prove that the soliton decays exponentially; concretely one obtains that if $v \geq 2\frac{\mu}{\nu}$ then for all β, with $0 \leq \beta < v^2 - (v - 2\frac{\mu}{\nu})^2$ is

$$\lim_{t\to\infty} \sqrt{2\pi t}\, e^{\frac{\beta}{2}}\left\langle \frac{\partial u}{\partial z}(t,z)\right\rangle = 0. \tag{38}$$

When $v < 2\frac{\mu}{\nu}$ (38) holds for all β, $0 \leq \beta < v^2$ and besides

$$\lim_{t\to\infty} \sqrt{2\pi t}\, e^{\frac{v^2 t}{2}}\left\langle \frac{\partial u}{\partial z}(t,z)\right\rangle = -2\frac{\mu^2}{\nu}e^{vz}\int dx\, e^{-vx}\mathrm{sech}^2\left[\frac{\mu}{\nu}x\right] = -2\frac{\nu v e^{vz}}{\sin\frac{\pi v \nu}{2\mu}}. \tag{39}$$

In contrast, along the trajectory (35) the decay is only rational

$$\lim_{t\to\infty} \sqrt{2\pi t}\left\langle \frac{\partial u}{\partial z}(t,z)\right\rangle = -2\frac{\mu^2}{\nu}e^{-\frac{x_0^2}{2}}\int dx\, \mathrm{sech}^2\left[\frac{\mu}{\nu}x\right] = -4\mu e^{-\frac{x_0^2}{2}}. \tag{40}$$

This means that due to the effect of noise the soliton spreads over a region with a width of order $O(\sqrt{t})$. Note that although formula (35) bears some resemblance to the dynamics of localized solutions of KPI equation [7, 8] the interpretation is totally different.

5. Burgers equation under other kinds of noise

Next we briefly consider Burger's equation (1), (2) coupled to an arbitrary noise $\zeta(t) \equiv \dot{\xi}$ defined by Ito's SDE

$$\frac{d\xi(t)}{dt} = a(t,\xi) + b(t,\xi)\frac{dW(t)}{dt}; \qquad \xi(0) = \xi_0 \tag{41}$$

where ξ_0 is a constant and $a, b : R^+ \times R \to R$ are given functions.

It is well known that the density $f_t(x)$ of $\xi(t)$ is given by

$$f_t(x) = f(t, x \mid \xi_0), \tag{42}$$

where $f(t, y\,|\,x)$ is a Green's function for the Fokker–Planck equations:

$$\frac{\partial f(t, y\,|\,x)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y^2}(b^2(t, y) f(t, y\,|\,x)) - \frac{\partial}{\partial y}(a(t, y) f(t, y\,|\,x));$$

$$\lim_{t\downarrow 0} f(t, y\,|\,x) = \delta(y - x). \tag{43}$$

Using Ito calculus one can prove, in a similar way to that described in Section 2, that the solution is given by $u(t, z) = g(t, z + \xi(t))$, where $g(t, z)$ solves

$$\frac{\partial g}{\partial t} + g\frac{\partial g}{\partial z} - \frac{\alpha^2 - b^2(t, z)}{2}\frac{\partial^2 g}{\partial z^2} = 0; \quad g(t = 0, z) = \psi(z - \xi_0). \tag{44}$$

The solution to the i.v.p (1), (2) is reduced to solving the SDE (41) and the nonlinear PDE (44). In general, we do not expect that (44) can be solved in a closed way unless $b(t, z)$ is a constant. In this case it reduces to either the shock-wave equation when $b = \alpha$, or to a standard Burgers equation when $b < \alpha$.

In this regard, a natural question is to determine the most general election of the functions $a(t, z)$ and $b(t, z)$ for which (41) is "integrable". Results in this direction have been obtained in [18]. We consider here the particular case corresponding to a generalized Ornstein–Uhlenbeck process or (colored) noise $\zeta(t) \equiv \dot{\xi}$ for which $a(t, y) = -\gamma(t)y$, $b(t, y) = 1$, $\xi_0 = 0$. Here $\gamma(t) > 0$ is a given function of time. One can solve (43), (44) to find

$$f(t, y\,|\,x) = \frac{1}{\sqrt{2\pi\,\Sigma^2}}\exp\left[-\frac{(y - xe^{-\Gamma(t)})^2}{\Sigma^2(t)}\right], \tag{45}$$

$$\Gamma(t) \equiv \int_{t_0}^{t}\gamma(s)\,ds; \quad \Sigma^2(t) \equiv e^{-2\Gamma(t)}\int_{t_0}^{t} e^{2\Gamma(s)}\,ds \tag{46}$$

and $u(t, z) = g(t, z + \xi(t))$ where $g(t, z)$ solves (4).

When γ is a constant we recover the classical Ornstein–Uhlenbeck process; unlike the B.M., this process *converges* to a Gaussian variable ξ_∞ with zero mean and variance $\frac{1}{2\gamma}$ as $t \to \infty$. It follows that $u(t, z)$ is also convergent to a well defined random variable $g(\infty, z + \xi_\infty)$. In particular one obtains the following. As $t \to \infty$ the kink $u(t, z)$ converges to a random variable:

$$u(t, z) \equiv v - 2\mu\tanh\left[\frac{\mu}{\nu}(z + \xi(t) - vt)\right]\underset{t\to\infty}{\to} v - 2\mu\tanh\frac{\mu}{\nu}[z - vt + \xi_\infty]. \tag{47}$$

Hence asymptotically the front wave has a random position $\tilde{z}_\infty(t) = vt - \xi_\infty$ which is normally distributed $\mathcal{N}(vt, \frac{1}{2\gamma})$, *but* the kink does not get destroyed. In addition, in the frame moving with the wave

$$\langle u_z(t, z)\rangle \to -\frac{2\mu^2}{\nu\sqrt{\pi}}\int dx\,\exp[-\gamma x^2]\mathrm{sech}^2\frac{\mu}{\nu}x \neq 0 \tag{48}$$

and

$$\langle u(t,z)\rangle \to v - \frac{2\mu\sqrt{\gamma}}{[\pi]^{1/2}} \int dx \exp[-\gamma x^2] \tanh \frac{\mu}{\nu} x. \tag{49}$$

The above analysis can be extended to general $\gamma(t)$ by evaluating the limit of the distribution (45). We find the following. If $l \equiv \lim_{t\to\infty} \gamma(t) = 0$, then $\xi(t)$ does not converge as $t \to \infty$ and neither will $u(t, z)$. If $0 < l < \infty$, $\xi(t)$ converges to a Gaussian variable $\xi_\infty = \mathcal{N}(0, \frac{1}{2l})$ and $u(t, z) \to g(t, z + \xi_\infty)$. Hence asymptotically the front wave of the kink has too a Gaussian position $\tilde{z}_\infty(t) = vt - \xi_\infty = \mathcal{N}(vt, \frac{1}{2l})$.

If $l = \infty$ then $\xi(t)$ converges to zero and $u(t, z)$ converges to the unperturbed solution: $u(t, z) \to g(t, z)$ located at $\tilde{z}_\infty(t) = vt$. This is remarkable; it means that if the dissipation term of the noise is sufficiently strong, the random wave will tend to the unperturbed solution $g(t, z)$.

Appendix

Let $X(t) \equiv W(t) + vt$ be a Brownian motion with drift coefficient v for which an absorbing barrier has been placed at the point $x = b$. If τ_b is the first passage time to b, this means that $X(t) = b$ for all $t \geq \tau_b$. Then it is known (see [16]) that its density $\tilde{f}(t, x)$ solves the following Dirichlet boundary problem:

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2} + v\frac{\partial}{\partial x}\right)\tilde{f}(t,x) = 0; \quad \lim_{t\downarrow 0} \tilde{f}(t,x) = \delta(x)$$

$$\tilde{f}(t,-\infty) = \tilde{f}(t, x = b) = 0.$$

To solve this equation define $\tilde{f}(t,x) = \exp(vx - \frac{v^2}{2}t)\psi(t,x)$ where we have

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\psi(t,x) = 0; \quad \psi(0,x) = \delta(x); \quad \psi(t,-\infty) = \psi(t,b) = 0.$$

Solving via Laplace transforms, it follows that

$$\psi(t,x) = [2\pi t]^{-1/2}\left[\exp\left(-\frac{x^2}{2t}\right) - \exp\left[-\frac{(x-2b)^2}{2t}\right]\right]$$

and

$$\tilde{f}(t,x) = [2\pi t]^{-1/2}\left[\exp\left[-\frac{(x-vt)^2}{2t}\right] - \exp\left[-\frac{(x-2b-vt)^2 - 4vtb}{2t}\right]\right].$$

Note next the obvious fact:

$$1 = P(X_t < b) + P(X_t = b) = \int_{-\infty}^{b} \tilde{f}(t,x)\,dx + P\{\tau_b \leq t\},$$

which implies by differentiation that

$$f_\tau(t) = -\int_{-\infty}^{b} \frac{\partial}{\partial t}\tilde{f}(t,x)\,dx = \int_{-\infty}^{b}\left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + v\frac{\partial}{\partial x}\right)\tilde{f}(t,x)\,dx$$

$$= v\tilde{f}(t,b) - \frac{1}{2}\frac{\partial \tilde{f}}{\partial x}(t,b) = [2\pi t^3]^{-1/2}|b|\exp\left[-\frac{(b-vt)^2}{2t}\right].$$

To determine the density of $M(t) \equiv \max_{t' \le t} z(t')$ we note that $M(t) \ge b \Leftrightarrow \tau_b \le t$; hence

$$P\{M \ge b\} = P\{\tau_b \le t\} = 1 - \int_{-\infty}^{b} \tilde{f}(t,x)\,dx.$$

If $f_M(b)$ stands for the relevant density we have that

$$f_M(b) \equiv -\frac{\partial}{\partial b}P\{M \ge b\} = -\int_{-\infty}^{b}\frac{\partial}{\partial b}\tilde{f}(t,x)\,dx.$$

The result follows after a tedious quadrature.

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