Theoretical and Mathematical Physics, **144**(2): 1132–1137 (2005)

SOLUTIONS OF A CAMASSA–HOLM HIERARCHY IN 2+1 DIMENSIONS

P. G. Estévez* and J. Prada†

We consider solutions of a generalization of the Camassa–Holm hierarchy to 2+1 *dimensions that include, in particular, the well-known multipeakon solutions for the celebrated Camassa–Holm equation.*

Keywords: Camassa–Holm equation, hodograph transformations, nonisospectral problems

1. Introduction

In 1993, Camassa and Holm found a completely integrable dispersive shallow-water equation, namely,

$$u_t + 2ku_x - u_{xxt} + 3uu_x - 2u_x u_{xx} + uu_{xxx} = 0, \tag{1.1}$$

where u is the fluid velocity in the x direction and k is a constant related to the critical shallow-water wave speed [1]. The limit $k = 0$ was given special attention in [2] because of its mathematical interest; in the search for solutions of the form $u(x,t) = U(x - ct)$ (U is a function that vanishes at infinity with its first and second derivatives), it was found that

$$U = ce^{-|x-ct|} + O(k \log k). \tag{1.2}$$

The form of traveling-wave solution (1.2) led Camassa and Holm to make the well-known solution ansatz for solutions with N interacting peaks of the Camassa–Holm (CH) equation.

Thereafter, much attention was given to peakon solutions. For instance, in 1994, Alber, Camassa, Holm, and Marsden investigated the geometry of peaked solitons for the general CH equation, i.e., Eq. (1.1), without assuming $k = 0$ [3]. The existence of a Liouville transformation mapping the CH spectral problem to the string problem was used by Beals, Sattinger, and Szmigielski in 1999 to present a closed form of the multipeakon solutions using a theorem of Stieltjes on continuous fractions [4]. The same authors investigated the relation between the multipeakons and the classical moment problem [5]. In 2000, Constantin and Strauss studied the stability of peakons [6], and Lenells studied the stability of periodic peakons [7] and presented a variational proof of it [8].

Later, in 2003, Degasperis, Holm, and Hone investigated an integrable equation with peakon solutions [9]. This equation, as the authors said, is similar to the CH shallow-water equation in form and was obtained by the method of asymptotic integrability. In its dispersionless form, it was written as

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \tag{1.3}$$

and it was proved that the single peakon

$$u(x,t) = ce^{-|x-ct|} \tag{1.4}$$

*Area de Fisica Teórica, Universidad de Salamanca, Salamanca 37008, Spain, e-mail: pilar@usal.es.
†Departamento de Matemáticas, Universidad de Salamanca, Salamanca 37008, Spain, e-mail: prada@usal.es.

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 144, No. 2, pp. 295–301, August, 2005.

is a solution. The authors also considered the family of equations

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx} \tag{1.5}$$

for the real parameter b [10], including both the CH equation ($b = 2$) and (1.3) ($b = 3$) as special cases. It turned out that all equations in the family have multipeakon solutions and satisfy a dynamical system that takes the Hamiltonian form only in the case $b = 2$. Hone and Wang isolated the peakon equations via the Wahlquist–Estabrook prolongation algebra method [11]. In [12], it was shown that the CH spectral problem gives two different integrable hierarchies of nonlinear evolution equations: one is of a negative-order CH hierarchy and the other is of a positive-order CH hierarchy. Moreover, it was seen that the celebrated CH equation is included in the negative-order CH hierarchy while the Dym-type equation is included in the positive-order CH hierarchy. Many papers deal with solutions of the CH hierarchy, for instance, [13].

In Sec. 2, we consider a generalization of the CH hierarchy to 2+1 dimensions [14], [15]. Following the lead of Camassa and Holm, we make an ansatz for the existence of certain multipeakon solutions in the different equations of the hierarchy and present the resulting dynamical system. This is naturally a PDE system, and the equations appearing are hence of different types: a set of equations involving derivatives with respect to the variable y, another set giving the recursive relations, and finally the evolution equations. In Sec. 3, we present examples of the dynamical systems in some particular cases. Finally, we list conclusions in Sec. 4.

2. The negative Camassa–Holm hierarchy in 2+1 dimensions

Consider the well-known negative Camassa–Holm hierarchy (NCHH) [12] for a field $u(x,t)$, i.e.,

$$u_t = R^{-n} u_x, \qquad R = J_0 J_1^{-1}, \tag{2.1}$$

where the hierarchy order n is an integer and J_0 and J_1 are the operators

$$J_0 = \partial^3 - \partial, \qquad J_1 = u\partial + \partial u, \qquad \partial = \frac{\partial}{\partial x}. \tag{2.2}$$

Introducing n functions $v_1(x,t), \dots, v_n(x,t)$ as in [15],

$$\begin{aligned} &v_1 = J_0^{-1} u_x \quad \Rightarrow \quad J_0 v_1 = u_x, \\ &v_k = J_0^{-1} J_1 v_{k-1} \quad \Rightarrow \quad J_0 v_k = J_1 v_{k-1}, \quad k = 2, \dots, n, \end{aligned} \tag{2.3}$$

we can write Eq. (2.1) as

$$u_t = J_1 v_n, \tag{2.4}$$

and can therefore consider the NCHH the $n+1$ Eqs. (2.3) and (2.4) in $n+1$ fields $u, v_1, \dots, v_n$.

Obviously, system (2.3), (2.4) reduces to the well-known CH equation for $n = 1$ [1].

2.1. Generalization to three dimensions. As was shown in [15], a simple generalization of system (2.3), (2.4) to three dimensions is the system

$$\begin{aligned} &U_y = J_0 V_1, \\ &J_0 V_k = J_1 V_{k-1}, \quad k = 2, \dots, n, \\ &U_t = J_1 V_n, \end{aligned} \tag{2.5}$$

where $U = U(x,t,y)$ and $V_j = V_j(x,t,y)$, which can be written as

$$U_t = R^{-n} U_y \tag{2.6}$$

and denoted by CHH(2+1). It is trivial to see that NCHH (2.1) is obtained from (2.6) by the reduction $\partial/\partial y = \partial/\partial x$.

2.2. Solutions of NCHH(2+1). We now make the ansatz for NCCH(2+1),

$$V_k(x,y,t) = \sum_{i=1}^{N} A_i^{k)}(y,t)\partial^{-1}(e^{-|x-q_i(y,t)|}) + \sum_{i=1}^{N} B_i^{k)}(y,t)e^{-|x-q_i(y,t)|}, \quad k = 1,2,\ldots,n, \tag{2.7}$$

where $A_i^{1)}(y,t) = 0$, $\forall i = 1,2,\ldots,N$, i.e.,

$$V_1(x,y,t) = \sum_{i=1}^{N} B_i^{1)}(y,t)e^{-|x-q_i(y,t)|} \tag{2.8}$$

is a multipeakon solution, and according to (2.5), U is

$$U(x,y,t) = -2\sum_{i=1}^{N} \gamma_i(y,t)\delta(x - q_i(y,t)). \tag{2.9}$$

Therefore, as in the CH equation, the peaks in V_1 are delta functions in U.

Substituting (2.7) and (2.9) in (2.5) yields the explicit formulation of the resulting system. We separate the three different sets of equations that appear in it: first, $2N$ equations involving derivatives with respect to the variable y; second, the recursive relations; third, the evolution equations. We have the following:

first,

$$\begin{aligned} &\frac{\partial \gamma_i(y,t)}{\partial y} = 0, \\ &\gamma_i(y,t)\frac{\partial q_i(y,t)}{\partial y} = -B_i^{1)}(y,t), \quad i = 1,2,\ldots,N; \end{aligned} \tag{2.10}$$

second,

$$\begin{aligned} A_i^{k)}(y,t) = {}& \sum_{j=1}^{N} \gamma_i(y,t)A_j^{k-1)}(y,t)e^{-|q_i(y,t)-q_j(y,t)|} - \\ & - \sum_{j=1}^{N} \gamma_i(y,t)B_j^{k-1)}(y,t)e^{-|q_i(y,t)-q_j(y,t)|}\operatorname{sgn}(q_i(y,t) - q_j(y,t)), \\ B_i^{k)}(y,t) = {}& -\sum_{j=1}^{N} \gamma_i(y,t)A_j^{k-1)}(y,t)(e^{-|q_i(y,t)-q_j(y,t)|} - 1)\operatorname{sgn}(q_i(y,t) - q_j(y,t)) + \\ & + \sum_{j=1}^{N} \gamma_i(y,t)B_j^{k-1)}(y,t)e^{-|q_i(y,t)-q_j(y,t)|}, \quad k = 2,\ldots,n, \quad i = 1,2,\ldots,N; \end{aligned} \tag{2.11}$$

third,

$$\frac{\partial\gamma_i(y,t)}{\partial t} = \gamma_i(y,t)\sum_{j=1}^{N} A_j^{n)}(y,t)e^{-|q_i(y,t)-q_j(y,t)|} -$$

$$- \gamma_i(y,t)\sum_{j=1}^{N} B_j^{n)}(y,t)e^{-|q_i(y,t)-q_j(y,t)|}\operatorname{sgn}(q_i(y,t)-q_j(y,t)),$$

$$\frac{\partial q_i(y,t)}{\partial t} = \sum_{j=1}^{N} A_j^{n)}(y,t)(e^{-|q_i(y,t)-q_j(y,t)|}-1)\operatorname{sgn}(q_i(y,t)-q_j(y,t)) -$$

$$- \sum_{j=1}^{N} B_j^{n)}(y,t)e^{-|q_i(y,t)-q_j(y,t)|}, \quad i=1,2,\ldots,N. \tag{2.12}$$

3. Examples in 2+1 dimensions

3.1. The case $n = 1$ and $N = 1$. We now consider the system

$$\begin{aligned} U_y &= (V_1)_{xxx} - (V_1)_x, \\ U_t &= 2U(V_1)_x + U_x V_1, \end{aligned} \tag{3.1}$$

which corresponds to the choice $n = 1$ in hierarchy (2.5). For $N = 1$, i.e., for

$$\begin{aligned} V_1(x,y,t) &= p(y,t)e^{-|x-q(y,t)|}, \\ U(x,y,t) &= -2\gamma(y,t)\delta(x-q(y,t)), \end{aligned} \tag{3.2}$$

we obtain

$$\begin{aligned} &\gamma(y,t) = \gamma_0, \\ &\frac{\partial q(y,t)}{\partial t} = \gamma_0\left(\frac{\partial q(y,t)}{\partial y}\right), \end{aligned} \tag{3.3}$$

i.e., $q(y,t) = F(y+\gamma_0 t)$, and therefore

$$\begin{aligned} V_1(x,y,t) &= -\frac{\partial q(y,t)}{\partial t}e^{-|x-q(y,t)|}, \\ U(x,y,t) &= -2\gamma_0\delta(x-q(y,t)). \end{aligned} \tag{3.4}$$

We note that if $q(y,t)$ has the form $q(y,t) = y+\gamma_0 t$, then a peakon solution of this system is given by

$$\begin{aligned} V_1(x,y,t) &= \gamma_0 e^{-|x-y-\gamma_0 t|}, \\ U(x,y,t) &= -2\gamma_0\delta(x-y-\gamma_0 t). \end{aligned} \tag{3.5}$$

3.2. The case $n = 1$ and $N = 2$: Two-soliton dynamics. We consider system (3.1) and take

$$\begin{aligned} V_1(x,y,t) &= p_1(y,t)e^{-|x-q_1(y,t)|} + p_2(y,t)e^{-|x-q_2(y,t)|}, \\ U(x,y,t) &= -2\gamma_1(y,t)\delta(x-q_1(y,t)) - 2\gamma_2(y,t)\delta(x-q_2(y,t)). \end{aligned} \tag{3.6}$$

Substituting these functions in (3.1) gives the dynamical system for the two-soliton case, from which we find that γ_1 and γ_2 are independent of y. Setting

$$F(t) = p_2(y,t)e^{|q_1(y,t)-q_2(y,t)|}\,\mathrm{sgn}(q_1(y,t) - q_2(y,t)), \tag{3.7}$$

$$G(t) = p_1(y,t)e^{|q_2(y,t)-q_1(y,t)|}\,\mathrm{sgn}(q_2(y,t) - q_1(y,t)), \tag{3.8}$$

we obtain

$$\log\gamma_1(t) = -\int F(t)\,dt, \qquad \log\gamma_2(t) = -\int G(t)\,dt, \tag{3.9}$$

and therefore

$$U(y,t) = -2e^{-\int F(t)\,dt}\delta(x - q_1(y,t)) - 2e^{-\int G(t)\,dt}\delta(x - q_2(y,t)), \tag{3.10}$$

where

$$\begin{aligned}
&\frac{\partial q_1(y,t)}{\partial y} = -e^{\int F(t)\,dt}p_1(y,t), \qquad \frac{\partial q_2(y,t)}{\partial y} = -e^{\int G(t)\,dt}p_2(y,t),\\
&\frac{\partial q_1(y,t)}{\partial t} = -p_1(y,t) - p_2(y,t)e^{-|q_1(y,t)-q_2(y,t)|},\\
&\frac{\partial q_2(y,t)}{\partial t} = -p_1(y,t)e^{-|q_2(y,t)-q_1(y,t)|} - p_2(y,t).
\end{aligned} \tag{3.11}$$

3.3. The case $n = 2$ and $N = 1$. For the system

$$\begin{aligned}
&U_y = (V_1)_{xxx} - (V_1)_x,\\
&(V_2)_{xxx} - (V_2)_x = 2U(V_1)_x + U_xV_1,\\
&U_t = 2U(V_2)_x + U_xV_2,
\end{aligned} \tag{3.12}$$

taking V_1 and U as in (3.2) and

$$V_2(x,y,t) = A_1^{2)}(y,t)\partial^{-1}(e^{-|x-q(y,t)|}) + B_1^{2)}(y,t)e^{-|x-q(y,t)|}, \tag{3.13}$$

we obtain

$$\begin{aligned}
&\frac{\partial\gamma(y,t)}{\partial y} = 0, \qquad \frac{\partial q(y,t)}{\partial y} = -\frac{p(y,t)}{\gamma(y,t)},\\
&A_1^{2)}(y,t) = 0, \qquad B_1^{2)}(y,t) = \gamma(y,t)p(y,t),\\
&\frac{\partial\gamma(y,t)}{\partial t} = 0, \qquad \frac{\partial q(y,t)}{\partial t} = -\gamma(y,t)p(y,t),
\end{aligned} \tag{3.14}$$

and therefore

$$\begin{aligned}
&\gamma(y,t) = \mathrm{const} = \gamma_0,\\
&V_1(x,y,t) = -\gamma_0\frac{\partial q(y,t)}{\partial y}e^{-|x-q(y,t)|},\\
&V_2(x,y,t) = -\frac{\partial q(y,t)}{\partial t}e^{-|x-q(y,t)|},\\
&U(x,y,t) = -2\gamma_0\delta(x - q(y,t)),\\
&\frac{\partial}{\partial t}(q(y,t)) = \gamma_0^2\frac{\partial q(y,t)}{\partial y},
\end{aligned} \tag{3.15}$$

where $q(y,t) = F(y + \gamma_0^2 t)$. We note that if $q(y,t) = y + \gamma_0^2 t$ in this case, then a solution of this system is

$$
\begin{aligned}
V_1(x,y,t) &= -\gamma_0 a e^{-|x-y-\gamma_0^2 t|}, \\
V_2(x,y,t) &= -\gamma_0^2 e^{-|x-y-\gamma_0^2 t|}, \\
U(x,y,t) &= -2\gamma_0 \delta(x - y - \gamma_0^2 t),
\end{aligned}
\tag{3.16}
$$

and both the variables V_1 and V_2 therefore have peakon solutions.

4. Conclusions

We have considered a generalization of the CH hierarchy to 2+1 dimensions and have studied certain solutions. Our main results are as follows:

1. We found the dynamical system and separated the different sets of equations involved in it: equations with derivatives with respect to the spatial variable y, the recursive relations, and the evolution equations.
2. We analyzed the resulting dynamical system in some particular cases and proved that peakon solutions exist.

Acknowledgments. This research was supported in part by the DGESYC (Contract Nos. BFM2002-02609 and BFM2003-00078) and also the Junta de Castilla y León (Contract No. SA011/04).

REFERENCES

1. R. Camassa and D. D. Holm, *Phys. Rev. Lett.*, **71**, 1661–166 (1993).
2. R. Camassa, D. D. Holm, and J. M. Hyman, *Adv. Appl. Mech.*, **31**, 1–33 (1994).
3. M. S. Alber, R. Camassa, D. D. Holm, and E. Marsden, *Lett. Math. Phys.*, **32**, 137–151 (1994).
4. R. Beals, D. H. Sattinger, and J. Szmigielski, *Adv. Math.*, **140**, 190–206 (1998); R. Beals, D. H. Sattinger, and J. Szmigielsji, *Inverse Problems*, **15**, L1–L4 (1999).
5. R. Beals, D. H. Sattinger, and J. Szmigielsji, *Adv. Math.*, **154**, 229–257 (2000).
6. A. Constantin and W. Strauss, *Comm. Pure Appl. Math.*, **53**, 603–610 (2000).
7. J. Lenells, *Int. Math. Res. Notices*, 485–489 (2004).
8. J. Lenells, *J. Nonlinear Math. Phys.*, **11**, 151–163 (2004).
9. A. Degasperis, D. D. Holm, and A. N. W. Hone, *Theor. Math. Phys.*, **133**, 1463–1474 (2002).
10. C. R. Gilson and A. Pickering, *J. Phys. A*, **28**, 7487–7494 (1995).
11. A. N. W. Hone and J. P. Wang, *Inverse Problems*, **19**, 129–145 (2003).
12. D. D. Holm and Z. Qiao, "The Camassa–Holm hierarchy, related N-dimensional integrable systems, and algebro-geometric solutions on a symplectic submanifold," nlin.SI/0202009 (2002).
13. F. Gesztesy and H. Holden, *Revista Mat. Iberoamericana*, **19**, No. 1, 73–142 (2003).
14. P. A. Clarkson, P. R. Gordoa, and A. Pickering, *Inverse Problems*, **13**, 1463–1476 (1997).
15. P. G. Estevez and J. Prada, *J. Phys. A*, **38**, 1287–1297 (2005); nlin.SI/0412019 (2004).