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The symmetry group of the quantum harmonic oscillator in an electric field

Research Article

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Abstract: In this paper we present two results. First, we derive the most general group of infinitesimal transformations for the Schrödinger Equation of the general time-dependent Harmonic Oscillator in an electric field. The infinitesimal generators and the commutation rules of this group are presented and the group structure is identified. From here it is easy to construct a set of unitary operators that transform the general Hamiltonian to a much simpler form. The relationship between squeezing and dynamical symmetries is also stressed. The second result concerns the application of these group transformations to obtain solutions of the Schrödinger equation in a time-dependent potential. These solutions are believed to be useful for describing particles confined in boxes with moving boundaries. The motion of the walls is indeed governed by the time-dependent frequency function. The applications of these results to non-rigid quantum dots and tunnelling through fluctuating barriers is also discussed, both in the presence and in the absence of a time-dependent electric field. The differences and similarities between both cases are pointed out.

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1. Introduction.

The Fermi accelerator is a simple but clever physical model [1] that as early as 1949 Fermi believed to be effective for explaining the origin of cosmic rays. Later on, Ulam and several other authors improved the original model to take into account other more sophisticated features not included in the original scheme [2, 3] and [4]. In its most simple form, the model considers a particle moving freely between two rigid walls. One of the walls remains at rest while the other oscillates with time under a not necessarily specified periodic law. This seemingly trivial ping-pong device has recently been applied to other phenomena with a surprisingly degree of success. Seba [5] used it as a model to explore chaos in quantum mechanics. The authors of the present work have previously reported [6, 7] a detailed study of the harmonic oscillator with time-dependent parameters, also using specific forms for this set of generalized variable frequencies. Other authors have elaborated more sophisticated applications based on the above model [8, 9] to describe theoretical models of Bose-Einstein condensation. The same approach has also been used in the characterization of quantum states of charged or neutral particles trapped in Penning [10] and Paul traps [11, 12], as well as in the description of the process of photon creation due to quantum fluctuations in cavities with moving walls [13] and in the characterization of metastable states in the presence of tunnelling with fluctuating barriers [14].

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Fostered by all of these really interesting physical effects, our aim here is to present a systematic study and an explicit exact construction of these new quantum states, of interest in systems in which time dependence plays a central role. The main tool used within the paper is the use of the maximal kinematic symmetry group of the generalized harmonic oscillator. Since the price to pay for introducing a time-dependent electric field is almost negligible, we have been able to consider such a case with almost no additional formal effort. After a complete analysis of the generalized symmetry group, we use it in a practical form to construct part of the huge set of exact wave functions that arise as a consequence of the set of generators of the group. Of special interest are those in which the time-dependent electric field coupled to the oscillator charge plays a dominant role.

Let us begin with the physical system described by the following Hamiltonian:

$$H(t) = \beta_3(t)\frac{p^2}{2m} + \frac{1}{2}\beta_2(t)\omega_o[p,x]_+ + \frac{1}{2}m\omega_o^2\beta_1(t)x^2 - qxE(t), \quad (1)$$

where $\{\beta_3(t), \beta_2(t)$ and $\beta_1(t)\}$ are real functions of time, q is the oscillator charge, and $E(t)$ is the external electric field. Notice that we have rearranged the Hamiltonian in such a way that all of these functions are dimensionless. Note also that one can easily achieve several limits by a judicious choice of the time-dependent functions: the free particle with or without time-dependent mass, the harmonic oscillator in the absence of electric field, etc. Furthermore, the Hamiltonian (1) also includes the case of a well with moving boundaries as it has been shown in [6] by the authors of the present work. These limits will also be carefully examined for each class of wave function obtained by the group theoretical method that we develop in Sections 2 and 3.

The Schrödinger equation arising from the Hamiltonian (1) in coordinate representation has the form:

$$i\hbar\psi_t = -\beta_3\frac{\hbar^2}{2m}\psi_{xx} - i\hbar\beta_2\omega_o\left(x\psi_x + \frac{1}{2}\psi\right) + \left(\frac{1}{2}m\omega_o^2\beta_1 x^2 - qEx\right)\psi, \quad (2)$$

where, as usual, $\psi(x,t)$ represents the wave function in the same representation. Before calculating the Symmetry Group of (2) in an appropiate way, we shall reduce the system to a more tractable form by using a unitary transformation. After this transformation, the symmetries and wave functions will be constructed in Section 3.

2. Reduction by unitary transformations

In this Section we shall follow the procedure described in Reference [15]. The aim is to reduce the initial Hamiltonian (1) to the form:

$$H_o(t) = i\hbar\dot{W}(t)W^+(t) + W(t)H(t)W^+(t) = \frac{p^2}{2m} + \frac{1}{2}m\Omega(t)^2x^2 \quad (3)$$

while the state vectors of both Hamiltonians are related as usual:

$$\mid\Psi(t)\rangle = W^+(t)\mid\Psi_o(t)\rangle, \quad (4)$$

where $\mid\Psi_o(t)\rangle$ is the state vector of the Hamiltonian $H_o(t)$ and $\mid\Psi(t)\rangle$ is the state vector of the Hamiltonian $H(t)$. Defining the effective frequency $\Omega^2(t)$ of (3) as

$$\Omega^2(t) = \omega_o^2\{\beta_1(t)\beta_3(t) - \beta_2^2(t)\} + \omega_o\left\{\beta_2(t)\frac{\dot{\beta}_3(t)}{\beta_3(t)} - \dot{\beta}_2(t)\right\} + \frac{\ddot{\beta}_3(t)}{2\beta_3(t)} - \frac{3\dot{\beta}_3^2(t)}{4\beta_3^2(t)} \quad (5)$$

direct calculation shows that the $W(t)$ operator appearing in (4) has the form:

$$W(t) = \exp\left\{-\frac{i}{2}qE_Z(t) - i\frac{q^2}{2m\hbar}\int_0^t\{E_u(s)dE_v(s) - E_v(s)dE_u(s)\}\right\}\exp\{i(f(t)p - m\dot{f}(t)x)\} \exp\left\{\frac{im}{2\hbar}(\omega_o\beta_2(t) - \frac{\dot{\beta}_3(t)}{2\beta_3(t)})x^2\right\}\exp\left\{\frac{i}{4\hbar}Log[\beta_3(t)][p,x]_+\right\} \quad (6)$$

with

$$f(t) = Z(t) + \frac{q}{m\hbar}(v(t)E_u(t) - u(t)E_v(t)), \quad (7a)$$

$$E_u(t) = \int_0^t\sqrt{\beta_3(s)}E(s)u(s)ds, \quad (7b)$$

$$E_v(t) = \int_0^t \sqrt{\beta_3(s)} E(s) v(s) ds \qquad (7c)$$

and $\{u(t), v(t)\}$ is a set of two independent solutions of the classical equations of motion corresponding to a harmonic oscillator of variable frequency $\Omega^2(t)$:

$$\ddot{u}(t) + \Omega^2(t) u(t) = 0,\ u(0) = 1,\ \frac{du(t)}{dt}|_{t=0} = \dot{u}(0) = 0, \qquad (8a)$$

$$\ddot{v}(t) + \Omega^2(t) v(t) = 0,\ v(0) = 0,\ \frac{dv(t)}{dt}|_{t=0} = \dot{v}(0) = 1. \qquad (8b)$$

We shall use $Z(t) = C_1 u(t) + C_2 v(t)$ to refer to an arbitrary linear combination of the main solutions $\{u(t), v(t)\}$. For $\Omega^2(t)$ real, one can consider with no loss of generality that $u(t)$ and $v(t)$ are also real. Thus, $Z(t)$ represents any classical solution of the equations of motion. To include the time-dependent electric field in the unitary transformation, we first define, in general:

$$E_\kappa = E_\kappa(t) = \int_0^t \sqrt{\beta_3(s)} E(s) \kappa(s) ds, \qquad (9)$$

where $\kappa(t)$ will usually be one of the functions: $\{u(t), v(t)\}$. Following the method described in [15, 16], one can easily generalize the $W(t)$ operator to include the time-dependent electric field. It takes the final form:

$$\begin{aligned} W(t) &= \exp\left\{-\frac{i}{2} q E_Z(t) - i\frac{q^2}{2m\hbar}\int_0^t \{E_u(s) dE_v(s) - E_v(s) dE_u(s)\}\right\} \\ &\cdot \exp\{\alpha(t) a^+ - \alpha^*(t) a\} \exp\left\{\beta(t)\frac{a^{+2}}{2} - \beta^*(t)\frac{a^2}{2}\right\} \exp\left\{i\frac{h(t)}{2}\left[a^+ a + \frac{1}{2}\right]\right\}, \end{aligned} \qquad (10)$$

where the $\alpha(t)$ appearing in the coherent state displacement operator is:

$$\alpha(t) = -\sqrt{\frac{\hbar m \omega_o}{2}}\left(f(t) + \frac{i}{\omega_o}\frac{df(t)}{dt}\right), \qquad (11)$$

$$f(t) = Z(t) + \frac{q}{\hbar m}\big(v(t) E_u(t) - u(t) E_v(t)\big) \qquad (12)$$

and the $h(t)$ and $\beta(t)$ appearing in the squeezed states generator are [15, 17]:

$$h(t) = 2\arctan\frac{\omega_o \beta_2(t) - \frac{\dot{\beta}_3(t)}{2\beta_3(t)}}{\omega_o(1 + \beta_3(t))}, \qquad (13)$$

$$\beta(t) = \arg\tanh\sqrt{\frac{\left[\omega_o\beta_2(t) - \frac{\dot{\beta}_3(t)}{2\beta_3(t)}\right]^2 + \omega_o^2[1 - \beta_3(t)]^2}{\left[\omega_o\beta_2(t) - \frac{\dot{\beta}_3(t)}{2\beta_3(t)}\right]^2 + \omega_o^2[1 + \beta_3(t)]^2}} \exp\left\{-i\arctan\left\{\frac{2\omega_o\left[\omega_o\beta_2(t) - \frac{\dot{\beta}_3(t)}{2\beta_3(t)}\right]}{\omega_o^2[\beta_3(t)^2 - 1] + \left[\omega_o\beta_2(t) - \frac{\dot{\beta}_3(t)}{2\beta_3(t)}\right]^2}\right\}\right\}. \qquad (14)$$

Up to a global phase factor, the electric field only enters the coherent states displacement operator but not the squeezing operator. The whole problem has thus been reduced to one of a time-dependent Hamiltonian operator:

$$H_o(t) = \frac{1}{2m}\{p^2 + m^2\Omega^2(t) x^2\}, \qquad (15)$$

with variable frequency given by:

$$\Omega^2(t) = \omega_o^2(\beta_1\beta_3 - \beta_2^2) + \omega_o\left(\beta_2\frac{\dot{\beta}_3}{\beta_3} - \dot{\beta}_2\right) + \frac{\ddot{\beta}_3}{2\beta_3} - \frac{3\dot{\beta}_3^2}{4\beta_3^2}, \qquad (16)$$

with a well-defined unitary transformation given by (3), (4) and (6)

3. The symmetry group

Here we shall follow the Olver formalism [18] to find vector fields associated with the symmetries of the Schrödinger equation arising from the Hamiltonian (15). These will all be of the form:

$$\hat{V} = \xi(x,t,\psi)\frac{\partial}{\partial x} + \tau(x,t,\psi)\frac{\partial}{\partial t} + \phi(x,t,\psi)\frac{\partial}{\partial \psi}, \quad (17)$$

with a second prolongation:

$$pr^{(2)}\hat{V} = \hat{V} + \phi^x\frac{\partial}{\partial \psi_x} + \phi^t\frac{\partial}{\partial \psi_t} + \phi^{xx}\frac{\partial}{\partial \psi_{xx}} + \phi^{xt}\frac{\partial}{\partial \psi_{xt}} + \phi^{tt}\frac{\partial}{\partial \psi_{tt}}. \quad (18)$$

Applying the formalism, we end up with a set of six vector fields:

$$\hat{V}_1 = u(t)\frac{\partial}{\partial x} + \frac{im}{\hbar}\dot{u}(t)x\psi\frac{\partial}{\partial \psi}, \quad (19a)$$

$$\hat{V}_2 = v(t)\frac{\partial}{\partial x} + \frac{im}{\hbar}\dot{v}(t)x\psi\frac{\partial}{\partial \psi}, \quad (19b)$$

$$\hat{V}_3 = u^2(t)\frac{\partial}{\partial t} + u(t)\dot{u}(t)x\frac{\partial}{\partial x} - \frac{1}{2}\{u(t)\dot{u}(t) + \frac{im}{\hbar}\{\Omega^2(t)u^2(t) - \dot{u}^2(t)\}x^2\}\psi\frac{\partial}{\partial \psi}, \quad (19c)$$

$$\hat{V}_4 = u(t)v(t)\frac{\partial}{\partial t} + \frac{1}{2}\frac{d\{u(t)v(t)\}}{dt}x\frac{\partial}{\partial x} - \left\{\frac{d\{u(t)v(t)\}}{dt} + \frac{2im}{\hbar}\{\Omega^2(t)u(t)v(t) - \dot{u}(t)\dot{v}(t)\}x^2\right\}\frac{\psi}{4}\frac{\partial}{\partial \psi}, \quad (19d)$$

$$\hat{V}_5 = v^2(t)\frac{\partial}{\partial t} + v(t)\dot{v}(t)x\frac{\partial}{\partial x} - \frac{1}{2}\left\{v(t)\dot{v}(t) + \frac{im}{\hbar}\{\Omega^2(t)v^2(t) - \dot{v}^2(t)\}x^2\right\}\psi\frac{\partial}{\partial \psi}, \quad (19e)$$

$$\hat{V}_6 = \psi\frac{\partial}{\partial \psi}. \quad (19f)$$

Since the only functions appearing in this set are $\Omega^2(t)$ and $\{u(t), v(t)\}$, this shows that the symmetry group of the reduced Hamiltonian can be found simply by using the time-dependent functions appearing in $H_o(t)$ and the set of linearly independent solutions $\{u(t), v(t)\}$ for the classical equations of motion. It is now interesting to consider the form of these generators when they act on the wave functions. The infinitesimal generators can be written in terms of x and p operators as:

$$\hat{V}_1 = \frac{i}{\hbar}\{u(t)p - m\dot{u}(t)x\}, \quad (20a)$$

$$\hat{V}_2 = \frac{i}{\hbar}\{v(t)p - m\dot{v}(t)x\}, \quad (20b)$$

$$\hat{V}_3 = -\frac{i}{2m\hbar}\{u^2(t)p^2 - mu(t)\dot{u}(t)\{xp+px\} + m^2\dot{u}(t)^2x^2\}, \quad (20c)$$

$$\hat{V}_4 = -\frac{i}{2m\hbar}\left\{u(t)v(t)p^2 - \frac{m}{2}\frac{d\{u(t)v(t)\}}{dt}\{xp+px\} + m^2\dot{u}(t)\dot{v}(t)x^2\right\}, \quad (20d)$$

$$\hat{V}_5 = -\frac{i}{2m\hbar}\{v^2(t)p^2 - mv(t)\dot{v}(t)\{xp+px\} + m^2\dot{v}(t)^2x^2\}, \quad (20e)$$

$$\hat{V}_6 = 1. \quad (20f)$$

The set of $\{v_1, ...v_6\}$ has already been reported recently in [19] and generalizes those described in [20] and [21] for the free particle case and the harmonic oscillator with constant frequency. The generators $\{v_1, ...v_6\}$ can be reduced to these particular cases: ($\Omega^2(t) = 0$ and $\Omega^2(t) = \omega_o^2$). They are antihermitian, although it is possible to improve the situation by constructing the function $\sigma(t)$:

$$\sigma(t) = u(t) + i\Omega_o v(t), \quad (21)$$

where Ω_o can be identified as the value $\Omega_o = \Omega(t=0)$. We now define the operators:

$$A = A(t) = \frac{1}{\sqrt{2m\hbar\Omega_o}}\{\sigma(t)p - m\dot{\sigma}(t)x\}, \quad (22)$$

$$A^+ = A^+(t) = \frac{1}{\sqrt{2m\hbar\Omega_o}}\{\sigma^*(t)p - m\dot{\sigma}^*(t)x\}. \quad (23)$$

It is easy to check that if $\{u(t), v(t)\}$ is a set of normalized linearly independent solutions of the classical equation of motion, we have:

$$[A(t), A^+(t)] = 1, \quad (24)$$

$$\frac{d}{dt}A(t) = \frac{\partial}{\partial t}A(t) + \frac{1}{i\hbar}[A(t), H_o(t)] = 0, \quad (25)$$

in such a way that the pair $A(t)$ and $A^+(t)$ are time-dependent invariants for $H_o(t)$. Let us construct a time-dependent realization of the above algebra:

$$\hat{V}_1 = i\sqrt{\frac{m\Omega_0}{2\hbar}}\{A(t) + A^+(t)\}, \quad (26a)$$

$$\hat{V}_2 = \sqrt{\frac{m}{2\hbar\Omega_0}}\{A(t) - A^+(t)\}, \quad (26b)$$

$$\hat{K}_+ = \hat{K}_+(t) = \frac{1}{2}\left\{\frac{i}{\Omega_0}\hat{V}_3 - i\Omega_0\hat{V}_5 + 2\hat{V}_4\right\} = \frac{1}{2}A^+(t)A^+(t), \quad (26c)$$

$$\hat{K}_- = \hat{K}_-(t) = \frac{1}{2}\left\{\frac{i}{\Omega_0}\hat{V}_3 - i\Omega_0\hat{V}_5 - 2\hat{V}_4\right\} = \frac{1}{2}A(t)A(t), \quad (26d)$$

$$\hat{K}_0 = \hat{K}_0(t) = \frac{i}{2}\left\{\frac{1}{\Omega_0}\hat{V}_3 + \Omega_0\hat{V}_5\right\} = \frac{1}{2}\left(A(t)^+A(t) + \frac{1}{2}\right). \quad (26e)$$

These five generators fulfill the following set of commutation rules:

$$[\hat{K}_0, \hat{K}_\pm] = \pm\hat{K}_\pm; \qquad [\hat{K}_+, \hat{K}_-] = -2\hat{K}_0, \quad (27a)$$

$$[\hat{V}_1, \hat{K}_+] = -\frac{1}{2}\{\hat{V}_1 + i\Omega_0\hat{V}_2\}; \;\; [\hat{V}_1, \hat{K}_-] = \frac{1}{2}\{\hat{V}_1 - i\Omega_0\hat{V}_2\}, \quad (27b)$$

$$[\hat{V}_2, \hat{K}_+] = \frac{1}{2}\{i\frac{\hat{V}_1}{\Omega_0} - \hat{V}_2\}; \;\; [\hat{V}_2, \hat{K}_-] = \frac{1}{2}\{i\frac{\hat{V}_1}{\Omega_0} + \hat{V}_2\}, \quad (27c)$$

$$[\hat{V}_1, \hat{K}_0] = \frac{i\Omega_0}{2}\hat{V}_2; \;\; [\hat{V}_1, \hat{V}_2] = -\frac{im}{\hbar}; \;\; [\hat{V}_2, \hat{K}_0] = -\frac{i}{2\Omega_0}\hat{V}_1. \quad (27d)$$

This is the Lie Algebra of $SU(1,1) \otimes \mathcal{H}_3$, where $\mathcal{H}_3$ represents the Heisenberg Algebra. The construction only holds for a vanishing electric field. It is not hard to extend the results to the case of non-vanishing electric fields, both constant and time-dependent. Let us now define the operators:

$$A_E(t) = \frac{\sqrt{\beta_3(t)}}{\sqrt{2m\hbar\Omega_o}}\left\{\sigma(t)p + m\frac{\sigma(t)}{\beta_3(t)}\left\{\omega_o\beta_2(t) - \frac{\dot{\beta}_3(t)}{2\beta_3(t)} - \frac{\dot{\sigma}(t)}{\sigma(t)}\right\}x - q\frac{E_u(t) + i\Omega_o E_v(t)}{\sqrt{\beta_3(t)}}\right\}, \quad (28)$$

$$A_E^+(t) = \frac{\sqrt{\beta_3(t)}}{\sqrt{2m\hbar\Omega_o}}\left\{\sigma^*(t)p + m\frac{\sigma^*(t)}{\beta_3(t)}\left\{\omega_o\beta_2(t) - \frac{\dot{\beta}_3(t)}{2\beta_3(t)} - \frac{\dot{\sigma}^*(t)}{\sigma^*(t)}\right\}x - q\frac{E_u(t) - i\Omega_o E_v(t)}{\sqrt{\beta_3(t)}}\right\}. \quad (29)$$

If $\{u(t), v(t)\}$ is a set of normalized linearly independent solutions of the classical equation of motion, the operators $A_E(t)$ and $A_E^+(t)$ also fulfill

$$[A_E(t), A_E^+(t)] = 1, \quad (30)$$

$$\frac{d}{dt}A_E(t) = \frac{\partial}{\partial t}A_E(t) + \frac{1}{i\hbar}[A_E(t), H(t)] = 0. \quad (31)$$

The operators built only with $A_E(t)$ and $A_E^+(t)$ are invariant. The operator $\frac{1}{2}(A_E^+(t)A_E(t) + \frac{1}{2})$ is also hermitian, and it belongs to the class of Lewis-Riesenfeld invariants for system (1). For a more detailed discussion of this type of operator, see [22] and [23] .

4. The wave functions

The unitary transformation $W(t)$ given by (9) can be used to construct the wave functions:

$$|\,\Psi(t)\rangle = W^+(t)\,|\,\Psi_o(t)\rangle, \quad (32)$$

or in terms of the coordinate representation:

$$\Psi(x,t) \;=\; \langle x \mid \Psi(t)\rangle = \langle x \mid W^{+}(t) \mid \Psi_o(t)\rangle = W^{+}(t)\Psi_o(x,t). \tag{33}$$

For an arbitrary function $\Xi(x,t)$, the following identities hold:

$$\exp\left\{\alpha(t)\frac{\partial}{\partial x}\right\}\Xi(x,t) = \Xi(x+\alpha(t),t), \tag{34}$$

$$\exp\left\{\alpha(t)(2x\frac{\partial}{\partial x}+1)\right\}\Xi(x,t) = e^{\alpha(t)}\Xi(e^{2\alpha(t)}x,t). \tag{35}$$

Making use of these, we finally find:

$$\Psi_E(x,t) \;=\; \frac{\exp\left\{-\frac{i}{2}\frac{q^2}{m\hbar}\int_0^t[E_v(s)dE_u(s)-E_u(s)dE_v(s)]\right\}}{\beta_3(t)^{\frac{1}{4}}}\exp\left\{\frac{i}{2}\frac{q}{\hbar}E_Z(t)-\frac{i}{4}\frac{m}{\hbar}\frac{\partial}{\partial t}\left[Z(t)-\frac{q}{m}g(t)\right]^2\right\} \tag{36}$$

$$\cdot\;\exp\left\{i\frac{m}{\hbar}\frac{x}{\sqrt{\beta_3(t)}}\left\{\dot{Z}(t)-\frac{q}{m}\dot{g}(t)-\frac{1}{2}\left[\omega_o\beta_2(t)-\frac{\dot{\beta}_3(t)}{2\beta_3(t)}\right]\frac{x}{\sqrt{\beta_3(t)}}\right\}\right\}\Xi\left\{\frac{x}{\sqrt{\beta_3(t)}}-Z(t)+\frac{q}{m}g(t),t\right\},$$

$$g(t) \;=\; u(t)E_v(t)-v(t)E_u(t), \tag{37}$$

with $\Xi(x,t)$ any wave function of the harmonic oscillator with variable frequency. The construction of the wave functions of the generalized oscillator **in the presence of a time-varying electric field** requires the solution of the following second-order differential equation.

$$i\hbar\frac{\partial}{\partial t}\Xi(y,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}\Xi(y,t)+\frac{1}{2}m\Omega(t)^2y^2\Xi(y,t). \tag{38}$$

To this end, we make use of the symmetries previously found in (17) [18]. After direct substitution, it is not hard to conclude that all the wave functions must have the general form:

$$\Xi(y,t) = \exp\left\{A(t)y^2+B(t)y+C(t)\right\}W(D(t)y+M(t)). \tag{39}$$

Now, $W(z)$ is the unknown function. Let us now impose that (39) should be a solution of the Schrödinger equation (38) and that W_o should be the Wronskian of the classical equation of motion. Defining:

$$Z(t) = C_1u(t)+C_2v(t), \tag{40a}$$

$$G(t) = Z(t)^2+C_5^2v(t)^2, \tag{40b}$$

$$Y(t) = C_3u(t)+C_4v(t). \tag{40c}$$

We have $W_o = C_1C_4 - C_2C_3$. We find the following non-trivial cases:

- For $C_1C_5 \neq 0$, there exists a solution for $W(z)$ fulfilling the ordinary second-order differential equation: $\ddot{W}(z) + (-\frac{z^2}{4}+\nu+\frac{1}{2})W(z) = 0$. The solution takes the form:

$$
\begin{aligned}
z(y,t) &= \sqrt{\frac{2C_1C_5 m}{\hbar}\frac{y+Y(t)}{\sqrt{G(t)}}}, \qquad (41)\\
\Xi_1(y,t) &= \frac{1}{G(t)^{\frac{1}{4}}}\exp\left\{i\left(\nu+\frac{1}{2}\right)\arctan\left[C_5\frac{C_3Z(t)-C_1Y(t)}{W_oZ(t)}\right]\right\}\\
&\cdot \exp\left\{\frac{im}{4\hbar}\frac{\left\{C_1C_5(W_o+C_3C_5)Y(t)-(W_o^2+C_3^2C_5^2)Z(t)\right\}^2}{C_1C_5W_o^2G(t)}\right\}\\
&\cdot \exp\left\{\frac{im}{\hbar}\left\{\frac{\dot{G}(t)}{G(t)}\frac{y^2}{4}-\sqrt{G(t)}\frac{\partial}{\partial t}\left\{\frac{Y(t)}{\sqrt{G(t)}}\right\}y\right\}\right\}\left\{\lambda_+\mathbf{D}_\nu\left\{z(y,t)\right\}+\lambda_-\mathbf{D}_\nu\left\{-z(y,t)\right\}\right\}, \qquad (42)
\end{aligned}
$$

where $\mathbf{D}_\nu(y)$ and $\mathbf{D}_\nu(-y)$ are the Weber functions. The Wronskian W_o can take an arbitrary constant value. As is well known, when $\nu=n$ is an integer, the solutions can be expressed in terms of Hermite polynomials:

$$
\begin{aligned}
\Xi_2(y,t) &= \frac{\lambda}{G(t)^{\frac{1}{4}}}\exp\left\{i\left(n+\frac{1}{2}\right)\arctan\left[C_5\frac{C_3Z(t)-C_1Y(t)}{W_oZ(t)}\right]\right\}\\
&\cdot \exp\left\{i\frac{m}{4\hbar}\frac{\left\{C_1C_5(W_o+C_3C_5)Y(t)-(W_o^2+C_3^2C_5^2)Z(t)\right\}^2+2iC_1^2C_5^2W_o^2Y(t)^2}{C_1C_5W_o^2G(t)}\right\}\\
&\cdot \exp\left\{i\frac{m}{\hbar}\left\{\frac{\dot{G}(t)+2iC_1C_5}{G(t)}\frac{y^2}{4}-\left\{\sqrt{G(t)}\frac{\partial}{\partial t}\left\{\frac{Y(t)}{\sqrt{G(t)}}\right\}-iC_1C_5\frac{Y(t)}{G(t)}\right\}y\right\}\right\}\mathbf{H}_n\left\{\frac{z(y,t)}{\sqrt{2}}\right\}. \qquad (43)
\end{aligned}
$$

- If $W_o=1$, there exists a solution for $W(z)$ fulfilling $\ddot{W}(z)-zW(z)=0$. The solutions can be expressed in terms of Airy functions:

$$
\begin{aligned}
z(y,t) &= \frac{y}{Y(t)}-\frac{\hbar^2}{4m^2}\frac{Z(t)^2}{Y(t)^2},\\
\Xi_3(y,t) &= \frac{1}{Y(t)^{\frac{1}{2}}}\exp\left\{i\frac{m}{2\hbar}\frac{\dot{Y}(t)}{Y(t)}y^2+i\frac{\hbar}{2m}\frac{Z(t)}{Y(t)^2}y-i\frac{\hbar^3}{12m^3}\frac{Z(t)^3}{Y(t)^3}\right\}\left\{\lambda_1\mathbf{Ai}(z(y,t))+\lambda_2\mathbf{Bi}(z(y,t))\right\}. \qquad (44)
\end{aligned}
$$

- If $W_o=1$, there also exists a solution in which $\ddot{W}(z)=0$. These solutions take the form:

$$
\Xi_4(y,t) = \frac{1}{Y(t)^{\frac{1}{2}}}\exp\left\{i\frac{m}{2\hbar}\frac{\dot{Y}(t)}{Y(t)}y^2-i\frac{y}{Y(t)}+i\frac{\hbar}{2m}\frac{Z(t)}{Y(t)}\right\}\left\{\lambda_1\left(\frac{y}{Y(t)}-\frac{\hbar}{m}\frac{Z(t)}{Y(t)}\right)+\lambda_2\right\}. \qquad (45)
$$

The dimensions of the constants are: $[C_1]=TL^{-1}$; $[C_2]=L^{-1}$; $[C_3]=L$; $[C_4]=LT^{-1}$; $[C_5]=L^{-1}$ and $[W_o]=1$ and they may be real or complex depending on the properties of the solutions.

One can always return to the original system by back-transforming these solutions in the way described above. We shall write the back-transformed solution only for the sake of completeness:

$$
\begin{aligned}
\Psi_E(x,t) &= \frac{1}{\beta_3(t)^{\frac{1}{4}}}\exp\left\{-\frac{i}{2}\frac{q^2}{m\hbar}\left\{\int_0^t[E_v(s)dE_u(s)-E_u(s)dE_v(s)]+g(t)\dot{g}(t)\right\}\right\}\\
&\cdot \exp\left\{-i\frac{q}{\hbar}\dot{g}(t)\frac{x}{\sqrt{\beta_3(t)}}-i\frac{m}{2\hbar}\left\{\omega_o\beta_2(t)-\frac{\dot{\beta}_3(t)}{2\beta_3(t)}\right\}\frac{x^2}{\beta_3(t)}\right\}\Xi\left\{\frac{x}{\sqrt{\beta_3(t)}}+\frac{q}{m}g(t),t\right\}, \qquad (46)\\
g(t) &= u(t)E_v(t)-v(t)E_u(t), \qquad (47)
\end{aligned}
$$

with $\Xi\{x,t\}$ any of the functions just found. Let us briefly consider certain particular cases of interest.

5. Particular cases of the wave functions

a) The free particle.

In this case, obviously $\beta_3(t) = 1$, $\beta_2(t) = 0$, $\beta_1(t) = 0$ and $E(t) = 0$. Also, and trivially, $u(t) = 1$, $v(t) = t$. The infinitesimal generators have been already described in [20].

• An initial judicious choice of the free parameters C_1, C_2, C_3 and C_4, taking into account their physical dimensions, leads us to the explicit solution:

$$\Psi(x,t) = \frac{1}{\sqrt{l_o + v_o t}} \exp\left\{ i\frac{m}{2\hbar}\frac{v_o x^2}{l_o + v_o t} + i\frac{p}{\hbar}\frac{l_o x}{l_o + v_o t} - i\frac{l_o p^2}{2\hbar m}\frac{t}{l_o + v_o t} \right\} \left(\lambda_1 \frac{m v_o x + l_o p}{l_o + v_o t} + \lambda_2 \right), \tag{48}$$

which reduces to the well-known plane wave solution for $v_o = 0$.

• With another set of free constants, we also find the normalized wave function:

$$\Psi_\alpha(x,t) = \left(\frac{m\Omega_o}{\pi\hbar}\right)^{\frac{1}{4}} \frac{\exp\left\{\frac{1}{4}(\alpha-\alpha^*)^2 - \alpha^2\right\}}{\sqrt{1+i\Omega_o t}} \exp\left\{ - \frac{\left(\sqrt{\frac{m\Omega_o}{2\hbar}}x - i\alpha\right)^2}{1+i\Omega_o t} \right\}, \tag{49}$$

where Ω_0 has dimensions of frequency and where we have chosen $\frac{C_4}{C_3} = I\Omega_o$. Below, we list the physical properties of this latter solution:

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\Omega_o}}\left\{i(\alpha-\alpha^*) + (\alpha+\alpha^*)\Omega_o t\right\}, \qquad \langle \Delta x\rangle = \sqrt{\frac{\hbar}{2m\Omega_o}}\sqrt{1+\Omega_o^2 t^2}, \tag{50a}$$

$$\langle p\rangle = \sqrt{\frac{m\hbar\Omega_o}{2}}(\alpha+\alpha^*), \qquad \langle \Delta p\rangle = \sqrt{\frac{m\hbar\Omega_o}{2}}, \tag{50b}$$

$$\langle E\rangle = \frac{1}{4}\left\{1+(\alpha+\alpha^*)^2\right\}\hbar\Omega_o, \qquad \langle \Delta E\rangle = \frac{1}{2}\hbar\Omega_o\sqrt{\frac{1}{2}+(\alpha+\alpha^*)^2}. \tag{50c}$$

This represents a Gaussian state that moves at constant velocity and with a probability density given by:

$$|\,\Psi_\alpha(x,t)\,|^2 = \frac{\sqrt{\frac{m\Omega_o}{\pi\hbar}}}{\sqrt{1+\Omega_o^2 t^2}} \exp\left\{ -\frac{m\Omega_o}{\hbar} \frac{\left\{x - \sqrt{\frac{\hbar}{2m\Omega_o}}\left\{i(\alpha-\alpha^*)+(\alpha+\alpha^*)\Omega_o t\right\}\right\}^2}{1+\Omega_o^2 t^2} \right\}. \tag{51}$$

This wave function represents a coherent state: an eigenstate of $A(t)$, with α the coherency parameter. It represents a statistical distribution of states with momentum p and a stationary momentum wavefunction given by:

$$\Phi_\alpha(p,t) = \frac{1}{(\pi m\hbar\Omega_o)^{\frac{1}{4}}} \exp\left\{\frac{1}{4}(\alpha-\alpha^*)^2\right\} \exp\left\{ -\left(\frac{p}{\sqrt{2m\hbar\Omega_o}} - \alpha\right)^2 \right\}. \tag{52}$$

• There also exist solutions of the free particle that use Airy functions [24]:

$$z(t) = \frac{l_1 + v_1 t}{l_o + v_o t}, \qquad v_1 l_o - l_1 v_o = \frac{\hbar}{m}, \tag{53}$$

$$\begin{aligned}\Psi(x,t) &= \frac{1}{\sqrt{l_o+v_o t}} \exp\left\{ i\frac{m}{2\hbar}\frac{v_o x^2}{l_o+v_o t} + \frac{i}{2}\frac{x}{l_o+v_o t}z(t) - \frac{i}{12}z(t)^3 \right\} \\ &\cdot \left\{ \lambda_1 \mathbf{Ai}\left\{\frac{x}{l_o+v_o t} - \frac{1}{4}z(t)^2\right\} + \lambda_2 \mathbf{Bi}\left\{\frac{x}{l_o+v_o t} - \frac{1}{4}z(t)^2\right\} \right\}.\end{aligned} \tag{54}$$

For $\lambda_2 = 0$, the wave function can be normalized on the positive real semiaxis.

• Finally, we shall construct normalizable states along the real axis for the free particle using the Weber functions or the Hermite polynomials for the integer index of the Weber functions. We now define $\frac{C_5}{C_1} = \Omega_o$ and $\frac{C_2}{C_1} = \Omega_1$, both with dimensions of frequency; also, $C_3 = 0$ and $C_4 = -\frac{P_o}{m}$, with dimensions of velocity. We obtain the following wave function:

$$\begin{aligned}\Psi_n(x,t) &= \left(\frac{m\Omega_o}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left\{\frac{-i(n+\frac{1}{2})\arctan\left\{\frac{\Omega_o t}{1+\Omega_1 t}\right\}\exp\left\{-\frac{P_o^2}{2m\hbar}\frac{\Omega_o}{\Omega_o^2+\Omega_1^2}\right\}}{\sqrt{2^n n!}(1+2\Omega_1 t+(\Omega_o^2+\Omega_1^2)t^2)^{\frac{1}{4}}}\right. \\ &\cdot \exp\left\{i\frac{m}{2\hbar}\frac{\Omega_1+i\Omega_o}{1+(\Omega_1+i\Omega_o)t}\left\{x+\frac{P_o}{m(\Omega_1+i\Omega_o)}\right\}^2\right\} \mathrm{H}_n\left\{\sqrt{\frac{m\Omega_o}{\hbar}}\frac{x-\frac{P_o}{m}t}{\sqrt{1+2\Omega_1 t+(\Omega_o^2+\Omega_1^2)t^2}}\right\}. \end{aligned} \tag{55}$$

These latter solutions have also been used to construct and characterize the in-phase entanglement of two particles. Any state of the free particle could in principle be described by an appropiate superposition of individual exact states. The ones now described are states with average position, momenta and energy, given by:

$$\langle x\rangle = \frac{P_o}{m}t, \qquad \Delta x = \sqrt{\frac{\hbar}{m\Omega_o}\left(n+\frac{1}{2}\right)(1+2\Omega_1 t+(\Omega_o^2+\Omega_1^2)t^2)}, \tag{56a}$$

$$\langle p\rangle = P_o, \qquad \Delta p = \sqrt{m\hbar\frac{\Omega_o^2+\Omega_1^2}{\Omega_o}\left(n+\frac{1}{2}\right)}, \tag{56b}$$

$$\langle E\rangle = \frac{P_o^2}{2m} + \hbar\frac{\Omega_o^2+\Omega_1^2}{2\Omega_o}\left(n+\frac{1}{2}\right), \qquad \Delta E = \hbar\frac{\Omega_o^2+\Omega_1^2}{2\Omega_o}\sqrt{\frac{2\Omega_o}{\Omega_o^2+\Omega_1^2}\frac{P_o^2}{\hbar m}(2n+1)+\frac{1}{2}(n^2+n+1)}. \tag{56c}$$

These expressions show that we are dealing with an equivalent system whose average energy equals that of a free particle plus a harmonic oscillator of constant frequency $\frac{\Omega_o^2+\Omega_1^2}{2\Omega_o}$. Note, however, that this is a dispersive wave in position, but that momentum and energy remain constant. This solution is physically equivalent to the representation of a collection of states with momentum distribution given by:

$$\begin{aligned}\Phi_n(p,t) &= \frac{\left\{-\frac{\Omega_o}{\pi m\hbar}\right\}^{\frac{1}{4}}\exp\left\{-i(n+\frac{1}{2})\arctan\frac{\Omega_o}{\Omega_1}\right\}}{(\Omega_o^2+\Omega_1^2)^{\frac{1}{4}}\sqrt{2^n n!}}\exp\left\{i\frac{\Omega_1 P_o^2}{2m\hbar(\Omega_1^2+\Omega_o^2)}\right\} \\ &\cdot \exp\left\{-i\frac{(p-P_o)^2}{2m\hbar(\Omega_1+i\Omega_o)}\right\}\mathrm{H}_n\left\{\sqrt{\frac{\Omega_o}{m\hbar}}\frac{p-P_o}{\sqrt{\Omega_o^2+\Omega_1^2}}\right\}.\end{aligned} \tag{57}$$

An analysis of the probability density of this wavefunction shows that we are really describing a vanishing state in the remote past and in the far future, while at the present time it is described by a Gaussian distribution that grows from zero and vanishes at longer times.

b) The well of moving walls.

By setting some infinite moving boudaries for the free particle one can obtain solutions for an infinite well with the walls moving under some time-dependent law. The general solution (39) allows the construction of such wave configurations. Indeed, the boundary conditions leads to:

$$\int_{a(t)}^{b(t)} |\,\Psi(x,t)\,|^2\,dx \quad < \quad \infty, \qquad \Psi(a(t),t) = \Psi(b(t),t) = 0, \tag{58}$$

where $a(t)$ and $b(t)$ are the bounded limits of the moving walls. We have found the following possible solutions under these conditions:

$$\Psi_n(x,t) = \sqrt{\frac{2}{(v_b-v_a)(t-t_o)}}\exp\left\{\frac{im}{2\hbar(t-t_o)}\left(x^2+\frac{n^2\pi^2\hbar^2}{m^2(v_b-v_a)^2}\right)\right\}\sin\left\{n\pi\frac{x-v_a(t-t_o)}{(v_b-v_a)(t-t_o)}\right\} \tag{59}$$

Here, $a(t) = v_a(t - t_o)$ and $b(t) = v_b(t - t_o)$. Therefore, the walls move with constant velocity and the width is a time-dependent function given by the law $L(t) = (v_b - v_a)(t - t_o)$. The case $v_a = 0$ is the particular one discussed in [25]. Other solutions constructed from (39) with similar boundary conditions are:

$$\Psi(x,t) = \frac{\lambda}{\sqrt{L(t)}} \exp\left\{-\frac{i\hbar^3}{24m^3}\frac{k_n^3}{(v_b - v_a)^3 L(t)}\left\{\frac{k_n^3}{L(t)^2} + 12\frac{m^2}{\hbar^2}v_a(v_b - v_a) - 3\frac{k_n^3}{(b-a)^2}\right\}\right\} \tag{60}$$

$$\cdot \exp\left\{\frac{im}{2\hbar}\frac{v_b - v_a}{L(t)}\left\{x - \frac{v_b a - v_a b}{v_b - v_a} + \frac{\hbar^2}{2m^2}\frac{k_n^3}{(v_b - v_a)^2}\left(\frac{1}{L(t)} - \frac{1}{b-a}\right)\right\}^2\right\}\left\{3^{-1/6}\mathbf{Ai}\left\{y(x,t)\right\} - 3^{-2/3}\mathbf{Bi}\left\{y(x,t)\right\}\right\},$$

$$y(x,t) = \frac{k_n}{L(t)}\left\{a + v_a t - \frac{\hbar^2}{4m^2(b-a)^2}\frac{k_n^3 t^2}{L(t)} - x\right\}. \tag{61}$$

The last solution represents the state of a particle in an infinite square well with moving boundaries of the form:

$$a(t) = a + v_a t - \frac{\hbar^2}{4m^2(b-a)^2}\frac{k_n^3 t^2}{L(t)}, \tag{62a}$$

$$b(t) = b + v_b t - \frac{\hbar^2}{4m^2(b-a)^2}\frac{k_n^3 t^2}{L(t)}, \tag{62b}$$

$$L(t) = b(t) - a(t) = b - a + (v_b - v_a)t. \tag{62c}$$

Quantization of momenta forces the k_n to be one of the roots of the transcendental equation:

$$\sqrt{3}\mathbf{Ai}[-k_n] - \mathbf{Bi}[-k_n] = 0. \tag{63}$$

This latter condition allows us to calculate the wavefunction and also the limits of the boundary, as stated above. For example, here if $v_a = v_b = v$, the function takes the simple form:

$$\Psi(x,t) = \lambda \exp\left\{\frac{2i}{3}\frac{m^3}{\hbar^3}\frac{(b-a)^3}{k_n^3}\left\{v - \frac{\hbar^2}{2m^2}\frac{k_n^3}{(b-a)^3}t\right\}^3\right\}\exp\left\{\frac{im}{\hbar}\left\{v - \frac{\hbar^2}{2m^2}\frac{k_n^3}{(b-a)^3}t\right\}\left\{x - a - \frac{m^2}{\hbar^2}\frac{(b-a)^3}{k_n^3}v^2\right\}\right\}$$

$$\cdot \left\{3^{-1/6}\mathbf{Ai}\left\{y(x,t)\right\} - 3^{-2/3}\mathbf{Bi}\left\{y(x,t)\right\}\right\}, \tag{64}$$

$$y(x,t) = \frac{k_n}{(b-a)}\left\{a + vt - \frac{\hbar^2}{4m^2}\frac{k_n^3}{(b-a)^3}t^2 - x\right\}, \tag{65}$$

which represents a well of constant width but with walls accelerating quadratically in time. The solutions for a moving wall with constant velocity were already described by [25]. The **accelerating moving wall** provided by the expressions (61) and (61) are a new and interesting result which can be checked by inserting these expressions in the Schrödinger equation.

Let us now work with the solutions of the free particle containing Hermite polynomials. After some algebraic calculations, we are able to set boundary conditions of the kind described above. The solutions and boundary laws now take the form:

$$\Psi_n(x,t) = \left(\frac{m\Omega_o}{\hbar}\right)^{\frac{1}{4}} \frac{\exp\left\{-\frac{m}{2\hbar}\frac{\Omega_o}{\Omega_o^2+\Omega_1^2}v^2\right\}}{\sqrt{\int_\alpha^\beta e^{-y^2}\mathbf{H}_n^2(y)dy}} \frac{\exp\left\{-i\left(n+\frac{1}{2}\right)\arctan\left\{\frac{\Omega_1}{\Omega_o} + \frac{\Omega_o^2+\Omega_1^2}{\Omega_o}t\right\}\right\}}{(1 + 2\Omega_1 t + (\Omega_o^2 + \Omega_1^2)t^2)^{\frac{1}{4}}}$$

$$\cdot \exp\left\{\frac{im}{2\hbar}\frac{\Omega_1 + i\Omega_o}{(1 + (\Omega_1 + i\Omega_o)t)}\left\{x - x_o + \frac{v}{\Omega_1 + i\Omega_o}\right\}^2\right\}\mathbf{H}_n\left\{\sqrt{\frac{m\Omega_o}{\hbar}}\frac{(x - x_o - vt)}{\sqrt{1 + 2\Omega_1 t + (\Omega_o^2 + \Omega_1^2)t^2}}\right\}, \tag{66}$$

which is normalized between limits, and changes with time in the form:

$$a(t) = \alpha\sqrt{\frac{\hbar}{m\Omega_o}}\sqrt{1+2\Omega_1 t+(\Omega_o^2+\Omega_1^2)t^2}+x_o+vt, \tag{67}$$

$$b(t) = \beta\sqrt{\frac{\hbar}{m\Omega_o}}\sqrt{1+2\Omega_1 t+(\Omega_o^2+\Omega_1^2)t^2}+x_o+vt, \tag{68}$$

with α and β two arbitrary roots of the Hermite polynomial H_n. Restricting ourselves to the states with odd index $(2n+1)$, we obtain:

$$\begin{aligned}\Psi_{2n+1}(x,t) &= \left(\frac{m\Omega_o}{\hbar}\right)^{\frac{1}{4}}\frac{\exp\left\{-i(2n+\frac{3}{2})\arctan\left\{\frac{\Omega_1}{\Omega_o}+\frac{\Omega_o^2+\Omega_1^2}{\Omega_o}t\right\}+i\frac{m}{2\hbar}\frac{\Omega_1+i\Omega_o}{1+(\Omega_1+i\Omega_o)t}x^2\right\}}{(1+2\Omega_1 t+(\Omega_o^2+\Omega_1^2)t^2)^{\frac{1}{4}}\sqrt{\int_0^\beta e^{-y^2}\mathbf{H}_{2n+1}^2(y)dy}}\\ &\cdot\ \mathbf{H}_{2n+1}\left\{\sqrt{\frac{m\Omega_o}{\hbar}}\frac{x}{\sqrt{1+2\Omega_1 t+(\Omega_o^2+\Omega_1^2)t^2}}\right\}.\end{aligned} \tag{69}$$

These are normalized states of the free particle in a square well with width starting at $a=0$ and variable length $L(t)=\beta\sqrt{\frac{\hbar}{m\Omega_o}}\sqrt{1+2\Omega_1 t+(\Omega_o^2+\Omega_1^2)t^2}$, where β is any root of the polynomial $\mathbf{H}_{2n+1}$. Also the expression (66) is a new and interesting result which can be checked by inserting this expression in the Schrödinger equation.

c) The harmonic oscillator with variable frequency.
In this case, obviously $\beta_3(t)=1$, $\beta_2(t)=0$, $\beta_1(t)=\frac{\Omega^2(t)}{\omega_o^2(t)}$ and $E(t)=0$. Also, in the case of an oscillator with constant frequency ω_o the characteristic functions are $u(t)=\cos(\omega_o t)$ and $v(t)=\frac{1}{\omega_o}\sin(\omega_o t)$. The infinitesimal generators have already been described in [21]. Some of the solutions have already been described in [6]. We shall proceed by discussing the most general case with variable frequency $\Omega^2(t)$. The first step is to find a pair of normalized solutions: $\{u(t),v(t)\}$ of the classical equations of motion. As we did in the free particle case, one can construct the wavefuntions simply with the help of $\{u(t),v(t)\}$ and the general expression described in Section 4. For the coherent class of solutions, we now obtain:

$$\begin{aligned}\Psi_\alpha(x,t) &= \left(\frac{m\Omega_o}{\pi\hbar}\right)^{1/4}\frac{\exp\left\{-\Im m[\alpha]^2\right\}}{\sqrt{u(t)+i\Omega_o v(t)}}\exp\left\{-\frac{i\Omega_o\alpha^2 v(t)}{u(t)+i\Omega_o v(t)}\right\}\\ &\cdot\ \exp\left\{\frac{im}{2\hbar}\frac{\dot{u}(t)+i\Omega_o\dot{v}(t)}{u(t)+i\Omega_o v(t)}x^2+\sqrt{\frac{2m\Omega_o}{\hbar}}\frac{i\alpha}{u(t)+i\Omega_o v(t)}x\right\}\end{aligned} \tag{70}$$

and for those containing Airy functions, we obtain:

$$\begin{aligned}\Psi(x,t) &= \frac{1}{\sqrt{l_1 u(t)+v_1 v(t)}}\exp\left\{i\frac{l_2 u(t)+v_2 v(t)}{l_1 u(t)+v_1 v(t)}\left\{\frac{2}{3}\frac{(l_2 u(t)+v_2 v(t))^2}{(l_1 u(t)+v_1 v(t))^2}-k\right\}\right\}\\ &\cdot\ \exp\left\{\frac{im}{2\hbar}\frac{l_1\dot{u}(t)+v_1\dot{v}(t)}{l_1 u(t)+v_1 v(t)}x^2-i\frac{l_2 u(t)+v_2 v(t)}{(l_1 u(t)+v_1 v(t))^2}x\right\}\{\lambda_1\mathbf{Ai}[y(x,t)]+\lambda_2\mathbf{Bi}[y(x,t)]\}\,,\end{aligned} \tag{71}$$

$$y(x,t) = \frac{x}{l_1 u(t)+v_1 v(t)}-\frac{(l_2 u(t)+v_2 v(t))^2}{(l_1 u(t)+v_1 v(t))^2}+k. \tag{72}$$

One can also construct wavefunctions for the Weber case, but we find it more useful to restrict ourselves to the case in which the Weber functions become Hermite polynomials owing to its broader physical applications. After some algebraic calculations we obtain:

$$\begin{aligned}\Psi_n(x,t) &= \left(\frac{m\Omega_o}{\pi\hbar}\right)^{\frac{1}{4}}\frac{\exp\left\{-i(n+\frac{1}{2})\arctan\frac{B(t)}{A(t)}\right\}\exp\left\{-\frac{m}{2\hbar}(\Omega_o x_o^2+i\frac{(v_o-(\Omega_1+i\Omega_o)x_o)^2}{\Omega_o}\frac{B(t)}{A(t)+iB(t)})\right\}}{\sqrt{2^n n!}\quad(A(t)^2+B(t)^2)^{\frac{1}{4}}}\\ &\cdot\ \exp\left\{i\frac{m}{2\hbar}\frac{\dot{A}(t)+i\dot{B}(t)}{A(t)+iB(t)}x^2+i\frac{m}{\hbar}\frac{v_o-(\Omega_1+i\Omega_o)x_o}{A(t)+iB(t)}x\right\}\mathbf{H}_n\left\{\sqrt{\frac{m\Omega_o}{\hbar}}\frac{x-F(t)}{\sqrt{A(t)^2+B(t)^2}}\right\},\end{aligned} \tag{73}$$

where:

$$F(t) = x_o u(t) + v_o v(t), \tag{74a}$$

$$A(t) = u(t) + \Omega_1 v(t), \tag{74b}$$

$$B(t) = \Omega_o v(t). \tag{74c}$$

The integration constants are defined as $C_3 = -x_o$ (an arbitrary length), $C_4 = -v_o$ (an arbitrary velocity) and arbitrary frequencies given by $\frac{C_5}{C_1} = \Omega_o$ and $\frac{C_2}{C_1} = \Omega_1$. Their position, momentum and energy averages are explicitly given by the expressions:

$$\langle x \rangle = x_o u(t) + v_o v(t), \qquad \Delta x = \sqrt{\frac{\hbar}{m\Omega_o}\left(n + \frac{1}{2}\right)(A(t)^2 + B(t)^2)}, \tag{75a}$$

$$\langle p \rangle = m(x_o \dot{u}(t) + v_o \dot{v}(t)), \qquad \Delta p = \sqrt{\frac{m\hbar}{\Omega_o}\left(n + \frac{1}{2}\right)(\dot{A}(t)^2 + \dot{B}(t)^2)}, \tag{75b}$$

$$\langle E \rangle = \frac{\langle p \rangle^2}{2m} + \frac{1}{2} m \langle x \rangle^2 \Omega(t)^2 + \frac{\hbar}{2\Omega_o}\left(n + \frac{1}{2}\right)\left\{(A(t)^2 + B(t)^2)\Omega(t)^2 + \dot{A}(t)^2 + \dot{B}(t)^2\right\}, \tag{75c}$$

$$\Delta E = \frac{\hbar}{2\Omega_o}\left\{\frac{1}{2}(1 + n + n^2)\left\{\left\{(A(t)^2 + B(t)^2)\Omega(t)^2 + (\dot{A}(t)^2 + \dot{B}(t)^2)\right\}^2 - 4\Omega_o^2\Omega(t)^2\right\}\right.$$
$$\left. + 4\frac{m\Omega_o}{\hbar}(1 + 2n)\left\{(A(t)^2 + B(t)^2)F(t)^2\Omega(t)^4 + 2\Omega(t)^2(A(t)\dot{A}(t) + B(t)\dot{B}(t))F(t)\dot{F}(t) + (\dot{A}(t)^2 + \dot{B}(t)^2)\dot{F}(t)^2\right\}\right\}^{\frac{1}{2}}. \tag{75d}$$

These states are a generalization of those described in [6]. Because the solutions do not admit separation of variables, they can be used to construct infinite quantum wells containing a quadratic potential by setting the boundary conditions at the frontier of the well. In this case the momentum distribution is:

$$F_n(p, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp\left\{-i\frac{p}{\hbar}(x - \frac{pt}{2m})\right\} \Psi_n(x, t) dx = \tag{76}$$
$$= \frac{\left(-\frac{\Omega_o}{\pi m\hbar}\right)^{\frac{1}{4}} \exp\left\{-i(n + \frac{1}{2})\arctan\frac{\dot{B}(t)}{\dot{A}(t)}\right\}}{\sqrt{2^n n!} \quad (\dot{A}(t)^2 + \dot{B}(t)^2)^{\frac{1}{4}}} \exp\left\{-\frac{m}{2\hbar}\left(\Omega_o x_o^2 + i\frac{(v_o - (\Omega_1 - i\Omega_o)x_o)^2}{\Omega_o}\frac{B(t) - t\dot{B}(t)}{A(t) + iB(t) - t(\dot{A}(t) + i\dot{B}(t))}\right)\right\}$$
$$\cdot \exp\left\{\frac{i}{2m\hbar}\left\{t - \frac{A(t) + iB(t)}{\dot{A}(t) + i\dot{B}(t)}\right\}\left\{p - \frac{m(v_o - (\Omega_1 - i\Omega_o)x_o)}{A(t) + iB(t) - t(\dot{A}(t) + i\dot{B}(t))}\right\}^2\right\} \mathrm{H}_n\left\{\sqrt{\frac{\Omega_o}{m\hbar}}\frac{p - m\dot{F}(t)}{\sqrt{\dot{A}(t)^2 + \dot{B}(t)^2}}\right\}$$

and the energy distribution takes the form:

$$C_k^n(\omega, t) = \int_{-\infty}^{\infty} \phi_k^*(x, t)\Psi_n(x, t) dx = \frac{(\omega\Omega_o)^{\frac{1}{4}}\sqrt{n!k!}}{2^{\frac{n+k-1}{2}}} \frac{(-1)^{\frac{n+k}{2}} \exp\left\{i(k + \frac{1}{2})\omega t\right\}}{\sqrt{\omega C(t) - i\dot{C}(t)}} \tag{77}$$
$$\cdot \exp\left\{-\frac{m}{2\hbar}\left(\Omega_o x_o^2 + i\frac{(v_o - (\Omega_1 + i\Omega_o)x_o)^2}{\Omega_o}\frac{\omega B(t) - i\dot{B}(t)}{\omega C(t) - i\dot{C}(t)}\right)\right\}\left\{\frac{\omega C^*(t) - i\dot{C}^*(t)}{\omega C(t) - i\dot{C}(t)}\right\}^{\frac{n}{2}}\left\{\frac{\omega C(t) + i\dot{C}(t)}{\omega C(t) - i\dot{C}(t)}\right\}^{\frac{k}{2}}$$
$$\cdot \sum_{j=0}^{\min[n,k]} \frac{(-1)^{\frac{n+j}{2}}}{j!(n-j)!(k-j)!}\left\{\frac{16\omega\Omega_o}{(\omega C(t) + i\dot{C}(t))(\omega C^*(t) - i\dot{C}^*(t))}\right\}^{\frac{j}{2}}$$
$$\cdot \mathrm{H}_{n-j}\left\{\sqrt{\frac{\Omega_o}{m\hbar}}\frac{m\omega\langle x \rangle - i\langle p \rangle}{\sqrt{(\omega C(t) - i\dot{C}(t))(\omega C^*(t) - i\dot{C}^*(t))}}\right\}\mathrm{H}_{k-j}\left\{\sqrt{\frac{m\omega}{\hbar}}\frac{v_o - (\Omega_1 + i\Omega_o)x_o}{\sqrt{(\omega C(t) + i\dot{C}(t))(\omega C(t) - i\dot{C}(t))}}\right\},$$

where $C(t) = A(t) + iB(t)$. The particular case of **constant frequency** $\Omega(t) = \Omega_o$ is obtained with the choice $\Omega_1 = 0$, $x_0 = 0$, $v_0 = 0$, and the pair of functions $u(t) = \cos(\Omega_o t)$ and $v(t) = \frac{1}{\Omega_o}\sin(\Omega_o t)$. For this well-known situation, in expression (77) one can carry out the summation exactly, which finally yields:

$$C_k^n(\omega, t) = \frac{(\omega\Omega_o)^{\frac{1}{4}}}{\sqrt{\omega+\Omega_o}} \frac{\sqrt{2k!}}{\sqrt{n!}} \exp\left\{i\left(k+\frac{1}{2}\right)\omega t - i\left(n+\frac{1}{2}\right)\Omega_o t\right\} \mathbf{P}^{\frac{n-k}{2}}_{\frac{n+k}{2}}\left[2\frac{\sqrt{\omega\Omega_o}}{\omega+\Omega_o}\right], \tag{78}$$

if $n+k$ is even, and zero otherwise. Indeed, $\mathbf{P}_l^m[x]$ denotes the associated Legendre polynomial:

$$\mathbf{P}_l^m[x] = (-1)^m \frac{(1-x^2)^{\frac{m}{2}}}{2^l l!} \frac{d^{m+l}}{dx^{m+l}}(x^2-1)^l. \tag{79}$$

It could be suggested that an arbitrary state with n modes in the typical frequency Ω_o oscillator can also be viewed as a superposition of states with k modes of another oscillator with arbitrary variable frequency ω. The statistical weight of each of these states in the former n-state would be $\mid C_k^n(\omega, t)\mid^2$.

d) The case with a non-vanishing time-dependent electric field.
So far we have considered systems with vanishing electric field. Let us now introduce a time-dependent electric field $E(t)$ coupled to the oscillator. Using the results of Section 2, the wavefunctions of the system with the electric field take the general form:

$$\Psi_E(x,t) = \exp\left\{-\frac{iq^2}{2m\hbar}\left\{\int_0^t g(s)E(s)ds + g(t)\dot{g}(t)\right\}\right\} \exp\left\{-\frac{iq}{\hbar}\dot{g}(t)x\right\} \Psi\left\{x+\frac{q}{m}g(t), t\right\}, \tag{80}$$

$$g(t) = u(t)E_v(t) - v(t)E_u(t), \tag{81}$$

with $E_u(t)$ and $E_v(t)$ as in (9) and $\Psi(y,t)$ any of the wavefunctions already analyzed in the previous Sections in the absence of an electric field. The mean values are now:

$$\langle x\rangle_E = \langle x\rangle_{E=0} - \frac{q}{m}\left\{u(t)E_v(t) - v(t)E_u(t)\right\}, \qquad \Delta x_E = \Delta x_{E=0}, \tag{82a}$$

$$\langle p\rangle_E = \langle p\rangle_{E=0} - q\left\{\dot{u}(t)E_v(t) - \dot{v}(t)E_u(t)\right\}, \qquad \Delta p_E = \Delta p_{E=0}. \tag{82b}$$

The momentum distribution can easily be related to that of the primitive (non-field) state. It now takes the form:

$$\begin{aligned}\Phi_E(p,t) = & \exp\left\{\frac{iq}{2m\hbar}\left\{2p + q(\dot{u}(t)E_v(t) - \dot{v}(t)E_u(t))\right\}\left\{E_v(t)(u(t)-t\dot{u}(t)) - E_u(t)(v(t)-t\dot{v}(t))\right\}\right\} \\ \cdot & \exp\left\{-\frac{iq^2}{2m\hbar}\int_0^t E(s)(u(s)E_v(s) - v(s)E_u(s))ds\right\} \Phi_{E=0}\left\{p + q[\dot{u}(t)E_v(t) - \dot{v}(t)E_u(t)], t\right\}.\end{aligned} \tag{83}$$

The photon distribution changes according to the initial wavefunction, $\Psi(y,t)$, chosen state. This is why it must be calculated in each case for this primitive wavefunction.

6. Conclusions and final remarks

Time-dependent harmonic oscillators and cavities with moving walls are of paramount importance for describing the physics of a few particles interacting with lasers in the nanoscale world. This has been shown to be of great importance in the study of tunnelling in the presence of fluctuating barriers [26]. Other examples of this are the study of quantum cascade lasers [27] and the description of quantum walls deformed by electrostatic potentials or even due to the plasticity effects that solids exhibit at small scale when confronted with higher-intensity fields of magnetic origin [28]. All of these effects are part of a growing branch of physics with excellent prospects in research ranging from pure quantum mechanics to industrial applications. The present paper, although originally quite mathematical, attempts to present a systematic study of the one-dimensional Schroödinger equation subjected to time-dependent harmonic potentials, with or without time-varying electric fields. First, a systematic study of the symmetry group for the harmonic oscillator with both time-dependent frequency and electric field is presented in order to classify the solutions later on. It is interesting to note that from the beginning this symmetry group exhibits an important set of features such as the coherent-state representation and the squeezing operators, both of which are widely used in quantum optics. We have also described a general method based on the symmetry group of the one-dimensional Schroödinger equation in order to construct the wave functions. This procedure later enables us to explore the different cases in detail and to provide the explicit physical properties of each, sometimes given in closed form. As a consequence, we have presented a general procedure for constructing the exact wave functions in the interacting system, starting from a well known-solution in the reduced case at zero external field. The striking result showing that the final expressions do not need to be of the separable-coordinate kind allows us to construct exact wave functions for quantum states confined in potential wells with moving walls and/or harmonic fluctuating potentials within elastic boundaries. The laws that govern the motion of the walls emerge as a consequence of the boundary conditions of each particular model describing the cavity.

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