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On the initial value problem for the KPII equation with data that do not decay along a line

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Abstract
We consider the Cauchy problem for the Kadomstev–Petviashvili II (KPII) equation with generic data that do not decay along a line. The linearization is in terms of spectral properties of the heat operator with a decaying 'time independent' potential. A bounded Green's function of this operator is constructed and its main properties are determined. The solution of the KPII equation is obtained via linear integral equations.

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1. Introduction

The Kadomtsev–Petviashili (KP) equation [1]

$$(u_t + u_{xxx} + 6uu_x)_x + \epsilon^2 3u_{yy} = 0 \tag{1}$$

is an extension to 2 + 1 dimensions of the Korteweg–deVries (KdV) equation and, like the latter in 1 + 1 dimensions, is the prototype multidimensional equation integrable by the inverse scattering transform (IST). It arises, for both $\epsilon^2 = 1$ or -1, in the study of two-dimensional surface water waves of small amplitude, which vary slowly in the direction transverse to that of wave propagation. It is also an important equation in plasma physics. Corresponding to $\epsilon^2 = -1$ we speak of KPI, while KPII corresponds to taking $\epsilon^2 = 1$. In [2–4] the relevant extension of IST to the multidimensional case was developed via a nonlocal Riemann problem and it was used to find the solution of the corresponding initial value problem (IVP) with decaying data for equation (1) with $\epsilon^2 = -1$; the solution to the IVP for KPII required new ideas that were set forth in [5] by considering a $\bar{\partial}$ problem. Subsequently, the Cauchy problem corresponding to decaying data for a number of important nonlinear evolution equations appearing in physics has been solved via IST with a mixture of the latter methods [6].

If the initial data do not satisfy the natural condition $\int u(x, y, 0)\,\mathrm{d}x = 0$, the KP evolution is, in principle, ill defined. This difficulty stems from the nonlocal nature of KP (in evolution form), and was first noticed in [7]. Corresponding to the linear problem, which, in this context, captures the key difficulties of the problem, the main ideas to overcome this problem were set forth in [8], where it was proved that time evolution exists, with the associated field decaying weakly as $t \to \infty$. It was also shown how, in this regard, the nonlinear problem is reducible to the linear one. Subsequently, interesting work was done in [9]; a complete rigorous study of the nonlinear Cauchy problem with unconstrained data was carried out in [10]. See also [11] for further discussion on this issue and the selection of boundary conditions.

Special KPI solutions are the lumps [12], which are localized configurations that decay rationally everywhere and possess simple dynamics. Recently, new localized solutions with rational decay but exhibiting nontrivial asymptotic dynamics have been found for KPI and several other integrable equations [13–15], while a spectral interpretation was given in [2, 16]. See [17, 18] for a discussion on the physical origin and integrability of KP [19, 20], for discussions on the rigorous theory and [11] for an updated account of all the above facts.

Unlike KPI, no localized multi-lump solutions are known to exist in KPII. However, the latter equation has line solutions: real and localized solutions that decay exponentially at infinity everywhere except along a line. From a dynamical perspective these are also asymptotically free objects moving with uniform velocities in which the only effect of interaction is a certain translation. We note that although line solitons were known as early as 1976 [21], they are not recovered by IST formulations in [2–6] as they are outside the class studied. In this regard, a natural problem is to generalize the results of [5] to the case with initial data that do not decay along a line, and, in particular, to recover the line solitons via IST. The generic solution to this problem has been elusive so far. Relevant previous work regarding the inverse scattering associated with the heat operator with a one soliton potential plus a decaying background appears in [22–25] (see also [26]) for a good general account of the above and the many difficulties inherent in this problem). For the time-dependent Schrödinger operator (SO) with one soliton potential plus decaying background and a KPI equation an important similar analysis has been performed in [27]. See also [28] in connection with other developments regarding, still, the pure one soliton case, and [29], where some results for the generic case of data that do not decay along a line were advanced.

Here, we present the IST solution to the IVP for the KPII equation with a generic data point which is real, nonsingular and is nondecaying along a line $\mathcal{L} = \{(x, y)|x - vy = x_0\}$. We decompose the initial data $u(x, y, 0) \equiv u(x, y)$ into an asymptotic nondecaying component along $\mathcal{L}$ and a decaying contribution: $u(x, y) = u_\infty(x - vy) + U(x, y)$, v constant, $u_\infty(x) \to 0$ as $|x| \to \infty$ and $U(x, y) \to 0$ as $r^2 = x^2 + y^2 \to \infty$; the linearization is done in terms of spectral properties of the heat operator $\hat{L} \equiv -\partial_y + \partial_{xx} + u_\infty(x)$ with a decaying 'time independent' potential $u_\infty(x)$ (see equation (3)). The relevant spectrum of $\hat{L}$ is determined by scattering data corresponding to the one-dimensional SO with potential $u_\infty(x)$ plus $\bar{\partial}$ data coming from the remaining potential $U(x, y)$.

The paper is organized as follows: in section 4 Green's function for the heat operator $\hat{L}$ satisfying

$$\hat{L}G \equiv (-\partial_y + \partial_{xx} + k^2 + u_\infty(x))G = -\delta(y)\delta(x - x')$$

is determined (see theorem 5). The construction is based on results of spectral theory and completeness relationships for an associated SO with potential $u_\infty(x)$. (For the classical theory of the SO the reader may consult [18, 30–32].) We establish the necessary results in this regard in section 3 (theorem 2 and proposition 3). Section 5 is devoted to the study of the main properties of the above Green's function. Propositions 6–8 yield the essential asymptotics for large values of the argument. We next study the properties of Green's function as a function of the complex spectral parameter and establish that, up to an exponential factor, it is bounded, decaying and continuous everywhere except at the set of points that comprise the discrete spectrum of the associated SO. The contribution of singularities to the $\bar{\partial}$ derivative of Green's function is twofold with both a pole and a delta function contribution appearing. We clarify how they affect the inverse problem (IP), and find that such singularities do not prevent the effective formulation of the direct and IPs. We elaborate on this in section 6. In theorem 13 we establish an integral equation for the direct problem and give sufficient conditions for solvability. Concretely, if the potentials satisfy

$$\int (1+|x|)u_\infty(x)\,\mathrm{d}x < \infty, |U(x,y)| < \infty, \qquad \int (1+x^2)|U(x,y)|\,\mathrm{d}x\,\mathrm{d}y < \epsilon,$$

where ϵ is a sufficiently small constant, we show that there exists a unique and bounded solution of the direct problem and that it is continuous everywhere except at a finite set of points $k \in \{k_j \equiv \mathrm{i}\kappa_j\}_{j=1,\ldots,N}$, which comprise the discrete spectrum of the associated SO, where the limits from any direction exist. The solution is finite and may increase linearly with $|x|$. Important properties are given in proposition 14.

The main result of the construction is established in theorem 15 wherein we establish, in terms of $\bar{\partial}$ and Riemann–Hilbert (RH) data, the integral equation for the IP with potentials that decay everywhere except along a line.

We next discuss the type of singularity that appears in the IP equations whenever a discrete spectrum of the associated SO is present. This difficulty was first noticed and discussed in [24, 25]. We show that the coefficient of the pole vanishes, thus smoothing the singularity.

In section 7 the time dependence of the data is found, whereupon, with IST, we linearize the Cauchy problem in the plane for equation (1) corresponding to a physical data point which is real, nonsingular and decays at infinity everywhere except along a line. The line solitons are recovered. In particular, our results imply (proposition 17) that line solitons are *nonlinearly stable* against generic decaying perturbations. This confirms well-known results (cf [1, 17] and [7] where the stability of these solitons against a particular kind of perturbation was suggested using a perturbative approach). We also note that the number of solitons that develop from KPII evolution with initial data $u(x,y,0) = u_\infty(x - vy) + U(x,y)$ is the same as the number of solitons that develop from KdV evolution with initial data $u(x,0) = u_\infty(x)$.

More general classes of line soliton solutions—moving along different lines—have been obtained by direct methods; the standard class was derived in [21], while other developments appear in [26, 33–40]. See also [41] for an interesting new class derived recently.

The important features of our results are as follow.

(i) Our results apply to generic potentials; in particular, line solitons may or may not be present.
(ii) The IP is a combination of a $\bar{\partial}$ contribution due to $U(x,y)$ coupled to a RH problem and pole contributions that are due to the spectrum arising from the SO with potential $u_\infty(x)$.
(iii) In the limit when u_∞ vanishes the solution of KPII reduces to the well-known $\bar{\partial}$ solution of [5], while if u_∞ corresponds to one soliton of the KPII equation our formalism recovers, in particular, the results of [24, 26].
(iv) The theory set forth here is not restricted to just the KP equation and carries over to other integrable equations. We also note that some of these results were outlined in [29], but proofs were not supplied.

2. Linearization

The linearization of KPII is connected [42] to the linear spectral equation $L\Psi(x, y) = 0$ where L is the heat operator defined as

$$L\Psi \equiv [-\partial_y + \partial_{xx} + u(x, y)]\Psi(x, y). \tag{2}$$

We assume that the initial data $u(x, y, t = 0)$ is real, nonsingular and decays at infinity everywhere except along a line $\mathcal{L}$ on the plane: $\mathcal{L} = \{(x, y)|x - vy = x_0\}$ (here x_0, v are the parameters defining the line), i.e. $\lim_{x,y\to\infty} u(x, y) = 0$ for $(x, y) \notin \mathcal{L}$ and if $(x, y) \in \mathcal{L}$, $\lim_{x,y\to\infty} u(vy + x_0, y) = u_\infty(x_0)$. Here, $u_\infty(x_0)$ is a given function that we assume is smooth and decaying: $\lim_{x_0\to\infty} u_\infty(x_0) = 0$. Consider a new function $\nu(x, y, k)$ via $\Psi(x, y) = \nu(x, y, k)\exp[-(v/2)x + (v^2/4 - k^2)y]$ that includes the spectral parameter k and define $U(x, y) \equiv u(x, y) - u_\infty(x_0)$. Note that $U(x, y)$ satisfies $\lim_{x,y\to\infty} U(x, y) = 0$. Finally, we consider new coordinates $x' \equiv x - vy$, $y' \equiv y$. In the new frame, and after dropping primes, we find that the spectral problem (2) associated with KPII reads

$$\hat{L}\nu(x, y, k) = -U(x, y)\nu(x, y, k); \qquad \hat{L} \equiv -\partial_y + \partial_{xx} + k^2 + u_\infty(x). \tag{3}$$

We consider a solution $\nu(x, y, k)$ to (3), which is defined for all values of the spectral parameter $k \in C$ and satisfies $\lim_{x^2+y^2\to\infty} \nu(x, y, k)e^{ikx} = 1$. To construct such a function, we need to define an appropriate Green's function for the heat operator with a 'time independent' potential $\hat{L}$. This construction is based on results on the spectral theory for the one-dimensional stationary SO. We take up these issues in the next section.

3. Completeness relations for the one-dimensional, stationary SO

Here, we develop the completeness relationships for eigenfunctions of the one-dimensional, stationary SO $A(x, \partial_x) \equiv \partial_{xx} + k^2 + u_\infty(x)$. These results are found to be critical to construct a Green's function for the heat operator $\hat{L}$ with a 'time independent' potential. We first recall several basic facts about the spectral theory of the former operator (the reader may consult [18, 30–32] regarding this). Define solutions $\phi_\pm(x, k)$, $\psi_\pm(x, k)$ to the equation

$$A(x, \partial_x)\phi(x, k) = A(x, \partial_x)\psi(x, k) = 0, \tag{4}$$

by requiring the conditions

$$\phi_\pm(x, k) = e^{\mp ikx}, \quad x \to -\infty; \qquad \psi_\pm(x, k) = e^{\pm ikx}, \quad x \to \infty$$

to hold. If $u_\infty(x)$ satisfies the condition

$$\int (1 + |x|)|u_\infty(x)|\,dx < \infty, \tag{5}$$

the former functions exist and are analytic functions of $k \equiv k_R + ik_I$ on $C_\pm$ (the upper/lower half k planes), having limits to the boundary $\{k_I = 0\}$. These limits satisfy the following relationship

$$\phi_+(x, k) = a(k)\psi_-(x, k) + b(k)\psi_+(x, k), \qquad k \in R, \tag{6}$$

for certain functions $a(k), b(k)$. $a(k)$ can be proved to be an analytic function of k on the upper half plane having a denumerable set $\{k_j \equiv i\kappa_j, \kappa_j \in R^+\}_{j=1,\dots,N}$ of (simple) zeros. If $\psi_j(x) \equiv \psi_+(x, k_j)$, then $\phi_+(x, k_j) = \beta_j\psi_j(x)$ for some complex constant β_j. The reflection coefficient $\rho(k) \equiv b/(a(k))$, the 'norming' constants β_j and the zeros k_j are the continuous and discrete scattering data of the one-dimensional SO.

Define $g_\pm(x,x',k)$ on $C_\pm$ as follows

$$g_+(x,x',k) \equiv \frac{\phi_+(x,k)\psi_+(x',k)}{a_+(k)}, \qquad k_I > 0, \tag{7.1}$$

$$g_-(x,x',k) \equiv -\frac{\phi_-(x',k)\psi_-(x,k)}{a_-(k)}, \qquad k_I < 0 \tag{7.2}$$

and

$$R(x,x',k) \equiv \frac{1}{a(k)}(\phi_+(x,k)\psi_+(x',k)\theta(x'-x) + \phi_+(x',k)\psi_+(x,k)\theta(x-x')). \tag{7.3}$$

We call $a_+(k) \equiv a(k)$, and note the symmetry conditions

$$a_-(\bar{k}) = -\bar{a}(k), \quad a_-(-k) = -a_+(k), \quad \phi_-(x,-k) = \phi_+(x,k), \quad \psi_-(x,-k) = \psi_+(x,k). \tag{8}$$

From the well-known results on one-dimensional scattering one has the following proposition.

Proposition 1. *Assume that the potential satisfies the condition (5). Then,*

(i) $R(x,x',k)$ *and* $g_+(x,x',k)$ *exist and are meromorphic functions on the upper half plane* C_+ *with poles at the numerable set* $\{k_j \equiv \mathrm{i}\kappa_j, \kappa_j \in R^+ : a(k_j) = 0\}_{j=1,\ldots,N}$*;* $g_-(x,x',k)$ *is meromorphic on the lower half plane* C_- *with poles at* $\{-k_j\}_{j=1,\ldots,N}$*; at these they have a residue*

$$\operatorname{Res} R(x,x',k)_{k=k_j} = \pm\operatorname{Res} g_\pm(x,x',k)_{k=\pm k_j} = C_j\psi_j(x)\psi_j(x') \equiv g_j(x,x');$$
$$C_j \equiv \frac{\beta_j}{a'(k_j)}. \tag{9}$$

(ii) As $|k| \to \infty$ *on the corresponding half plane,* $R(x,x',k)$ *and* $g_\pm(x,x',k)$ *have the asymptotic expansion*

$$R(x,x',k) = \mathrm{e}^{\mathrm{i}k|x-x'|}\left(1 + \sum_{n=1}^{\infty}\frac{R_n(x,x')}{k^n}\right); \qquad g_\pm(x,x',k) = \mathrm{e}^{-\mathrm{i}k(x-x')}\tilde{g}_\pm(x,x',k)$$
$$\text{with } \tilde{g}_\pm(x,x',k) \equiv \left(1 + \sum_{n=1}^{\infty}\frac{m_{n\pm}(x,x')}{k^n}\right), \tag{10}$$

where the coefficients are uniformly bounded.

We next obtain an important result for what follows.

Theorem 2. *For eigenfunctions of the SO with potential satisfying (5), the completeness relation below holds*

$$\int g_+(x,x',k)\,\mathrm{d}k \equiv \int \frac{\mathrm{d}k}{a(k)}\phi_+(x,k)\psi_+(x',k) = 2\pi\delta(x-x') + 2\pi\mathrm{i}\sum_j g_j(x,x') \tag{11.1}$$

and

$$\int g_-(x,x',k)\,\mathrm{d}k \equiv \int \frac{-\mathrm{d}k}{a_-(k)}\phi_-(x',k)\psi_-(x,k) = 2\pi\delta(x-x') + 2\pi\mathrm{i}\sum_j g_j(x,x'), \tag{11.2}$$

where, at infinity, a principal value prescription is taken.

Proof. Let Γ be the standard semicircular half-arc contour on the upper half plane going in the positive (anticlockwise) direction. In view of the properties (i) and (ii) of $R(x, x', k)$ one has

$$\int_\Gamma \mathrm{d}k(R(x, x', k) - \mathrm{e}^{\mathrm{i}k|x-x'|}) = 2\pi\mathrm{i}\sum_j g_j(x, x').$$

Using Jordan's lemma one also has that with the integrals taken in a distributional setting

$$\begin{aligned}\int_\Gamma \mathrm{d}k(R(x, x', k) - \mathrm{e}^{\mathrm{i}k|x-x'|}) &= \int_{-\infty}^{\infty}(R(x, x', k) - e^{\mathrm{i}k|x-x'|})\,\mathrm{d}k\\ &= \int_{-\infty}^{\infty} R(x, x', k)\,\mathrm{d}k - 2\pi\delta(x - x').\end{aligned}$$

Thus, one has that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} R(x, x', k)\,\mathrm{d}k = \delta(x - x') + \mathrm{i}\sum_j g_j(x, x').$$

Note next that

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{1}{a(k)}[\phi_+(x, k)\psi_+(x', k) - \phi_+(x', k)\psi_+(x, k)]\,\mathrm{d}k\\ &\quad= \int_{-\infty}^{\infty}[(\psi_-(x, k) + \rho(k)\psi_+(x, k))\psi_+(x', k)\\ &\qquad-(\psi_-(x', k) + \rho(k)\psi_+(x', k))\psi_+(x, k)]\,\mathrm{d}k\\ &\quad= \int \mathrm{d}k(\psi_-(x, k)\psi_+(x', k) - \psi_-(x', k)\psi_+(x, k)\\ &\quad= \int \mathrm{d}k(\psi_-(x, k)\psi_+(x', k) - \psi_+(x', k)\psi_-(x, k) = 0.\end{aligned}$$

This implies (11.1).

To prove the second relation note that

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{1}{a(k)}\phi_+(x, k)\psi_+(x', k)\,\mathrm{d}k &= \int_{-\infty}^{\infty}\frac{1}{a(k)}\phi_+(x', k)\psi_+(x, k)\,\mathrm{d}k\\ &= -\int_{-\infty}^{\infty}\frac{1}{a_-(k)}\phi_-(x', k)\psi_-(x, k)\,\mathrm{d}k. \qquad \square\end{aligned}$$

Another interesting development in this regard is the following theorem.

Theorem 3. *Let $k \in C$ and let $C'_\pm$ be the contour along the real axis with a small semicircular indentation below (+)/above (−) the point $l = -k_R$, which corresponds to the zeros of $a(l + k) = 0$ when $k = k_j = \mathrm{i}\kappa_j$, and such that at infinity a principal value prescription is taken. Then, for eigenfunctions of the SO with potential $u_\infty(x)$ satisfying (5), the completeness relation below holds*

$$\frac{1}{2\pi}\int_{C'_\pm} g_\pm(x, x', l + k)\,\mathrm{d}l = \delta(x - x') + \mathrm{i}\sum_{|k_I|\leqslant\kappa_j} g_j(x, x'). \tag{12}$$

Proof. If the symbol $|_{C'_\pm}$ means that we indent the contour as described above we have that the left-hand side of (12) is

$$\frac{1}{2\pi}\lim_{R\to\infty}\int_{-R+\mathrm{i}k_I}^{R+\mathrm{i}k_I} g_\pm(x, x', l)\,\mathrm{d}l|_{C'_\pm}.$$

We consider the case of g_+. Using the analyticity properties (9) and (10) of $g_\pm(x, x', l)$ we can deform the given contour to the real axis. In doing so, we pick a contribution from the poles within the contour, and obtain

$$\frac{1}{2\pi}\lim_{R\to\infty}\int_{-R+\mathrm{i}k_I}^{R+\mathrm{i}k_I} g_+(x,x',l)\,\mathrm{d}l\bigg|_{C'_+} = \frac{1}{2\pi}\lim_{R\to\infty}\int_{-R}^{R}\mathrm{d}l\,g_+(x,x',l) - \mathrm{i}\sum_{\kappa_j<k_I} g_j(x,x')$$

$$= \delta(x-x') + \mathrm{i}\sum_j g_j(x,x') - \mathrm{i}\sum_{\kappa_j<k_I} g_j = \delta(x-x') + \mathrm{i}\sum_{k_I\leqslant\kappa_j} g_j. \qquad \square$$

Proposition 4. *Let $h(k)$ be an even function on the line. Then,*

$$\int_{-\infty}^{\infty} h(k)g_+(x',x,k)\,\mathrm{d}k = \int_{-\infty}^{\infty} h(k)g_+(x,x',k)\,\mathrm{d}k = \int_{-\infty}^{\infty} h(k)g_-(x,x',k)\,\mathrm{d}k$$
$$= \int_{-\infty}^{\infty} h(k)g_-(x',x,k)\,\mathrm{d}k. \tag{13}$$

Proof.

$$\int_{-\infty}^{\infty} h(k)g_+(x',x,k)\,\mathrm{d}k = \int \frac{h(k)}{a(k)}\phi_+(x',k)\psi_+(x,k)\,\mathrm{d}k$$
$$= -\int_{-\infty}^{\infty}\frac{h(k)}{a_-(k)}\phi_-(x',k)\psi_-(x,k)\,\mathrm{d}k = \int_{-\infty}^{\infty} h(k)g_-(x,x',k)\,\mathrm{d}k.$$

In addition,

$$\int_{-\infty}^{\infty} h(k)[g_+(x,x',k) - g_+(x',x,k)]\,\mathrm{d}k = \int_{-\infty}^{\infty} h(k)[(\psi_-(x,k)+\rho(k)\psi_+(x,k))\psi_+(x',k)$$
$$-(\psi_-(x',k)+\rho(k)\psi_+(x',k))\psi_+(x,k)]\,\mathrm{d}k$$
$$= \int \mathrm{d}k\,h(k)(\psi_-(x,k)\psi_+(x',k) - \psi_-(x',k)\psi_+(x,k))$$
$$= \int \mathrm{d}k\,h(k)(\psi_-(x,k)\psi_+(x',k) - \psi_+(x',k)\psi_-(x,k)) = 0. \qquad \square$$

4. Green's function for the heat operator with a 'time-independent' potential

Here, we show how to construct a Green's function for the heat operator $\hat{L}$ of (3) with a 'time independent' potential. Let

$$G_c(x,x',y,k) = G_{c+}(x,x',y,k)\theta(k_I) + G_{c-}(x,x',y,k)\theta(-k_I),$$

where

$$G_{c\pm} = \frac{1}{2\pi}\int_{C'_\pm} \mathrm{e}^{-y(l^2+2kl)} g_\pm(x,x',l+k).$$
$$[\theta(y)(\chi_{A_+}\theta(k_R) + \chi_{A_-}\theta(-k_R)) - \theta(-y)(\chi_{A_+^c}\theta(k_R) + \chi_{A_-^c}\theta(-k_R)]\,\mathrm{d}l, \tag{14}$$

where

$$A_+ \equiv (-\infty, -2k_R] \cup (0,\infty); \qquad A_- \equiv (-\infty, 0] \cup (-2k_R, \infty), \tag{15}$$

$A^c_\pm$ stands for: $A^c_+ \equiv R - A_+ = (-2k_R, 0]$, $A^c_- \equiv R - A_- = (0, -2k_R]$, the indicator function χ_A of the set $A \subset R$ is defined as

$$\chi_A(l) = 1, \quad l \in A, \qquad \text{and} \qquad \chi_A(l) = 0 \quad \text{if } l \notin A,$$

$C'_\pm$ is a contour along the real axis with a small semicircular indentation below (+)/above (−) the point $l = -k_R$, and the symbol $|_{C'_\pm}$ means that we indent the contour as described above; as $|l| \to \infty$ a principal value prescription is taken.

The discrete part of Green's function is taken as

$$G_d(x, x', y, k) = \mathrm{i} \sum_{|k_j| \geqslant |k_I|} \mathrm{e}^{(k^2+\kappa_j^2)y} g_j(x, x')\theta(-y). \tag{16}$$

Finally, Green's function

$$G(x, x', y, k) = G_+(x, x', y, k)\theta(k_I) + G_-(x, x', y, k)\theta(-k_I)$$

is taken to be

$$G(x, x', y, k) = G_c(x, x', y, k) + G_d(x, x', y, k). \tag{17}$$

Theorem 5. *$G(x, x', y, k)$ is a Green's function for $\hat{L}$:*

$$\hat{L}G \equiv (-\partial_y + \partial_{xx} + k^2 + u_\infty(x))G = -\delta(y)\delta(x - x'). \tag{18}$$

Proof. By direct derivation and using theorem 3 we find that

$$\hat{L}G_{c\pm} = \frac{1}{2\pi} \int_{C'_\pm} [\theta(y)(\chi_{A_+}\theta(k_R) + \chi_{A_-}\theta(-k_R)) - \theta(-y)(\chi_{A_+^c}\theta(k_R) + \chi_{A_-^c}\theta(-k_R)].$$

$$\mathrm{e}^{-y(l^2+2kl)}[\partial_{xx} + (k + l)^2 + u_\infty(x)]g_\pm(x, x', l + k)\,\mathrm{d}l - \frac{\delta(y)}{2\pi} \int_{C'_\pm} g_\pm(x, x', l + k)\,\mathrm{d}l$$

$$= -\delta(y) \left(\delta(x - x') + \mathrm{i} \sum_{\kappa_j \geqslant |k_I|} g_j(x, x') \right).$$

Likewise,

$$\hat{L}G_d = \mathrm{i}\delta(y) \sum_{\kappa_j \geqslant |k_I|} g_j(x, x'). \qquad \square$$

5. Properties of Green's function

We next establish the main properties of Green's function. They will be used when we consider the solution of the IP.

Note the 'asymmetry' in the definition of $G_c(x, x', y, k)$, which has different definitions on the upper and lower half k-planes (see equation 14). Note also that the integrand in Green's function has pole singularities when $k_I = \pm\kappa_j$. This indicates that G might be discontinuous at both the real k-axis and the lines $k_I = \pm\kappa_j$. In spite of this, one has the following proposition.

Proposition 6. *On the imaginary axis, $k_R = 0$.*

(i) Green's function satisfies

$$G_+(x, x', y, \mathrm{i}k_I) = G_-(x, x', y, -\mathrm{i}k_I). \tag{19}$$

(ii) If the potential $u_\infty(x)$ satisfies (5) and $k_I \neq \pm\kappa_j$, Green's function vanishes exponentially fast as either $|y|$ or $|x|$ tend to ∞; concretely, we obtain the following.

As $y \to \infty$ with $(x-x')/(2y) \equiv l_0$ fixed $G_{c\pm}(x, x', y, k = \pm \mathrm{i}k_I)$ has an asymptotic expansion with leading term given by

$$
\begin{aligned}
G_{c\pm}(x, x', y, \pm\mathrm{i}k_I) \approx{}& \mathrm{e}^{-(k_I^2+l_0^2)y}\frac{g_\pm(x, x', -\mathrm{i}l_0)}{\sqrt{4\pi y}}\theta(y) \\
&+\mathrm{i}\left(\sum_{|k_j|\leqslant|l_0|} - \sum_{|k_j|<|k_I|}\right) g_j(x, x')\mathrm{e}^{(\kappa_j^2-k_I^2)y}\theta(y).
\end{aligned} \tag{20}
$$

As $|x-x'|/y \to \infty$ with $y \neq 0$ fixed $G_{c\pm}(x, x', y, k = \pm\mathrm{i}k_I)$ has an asymptotic expansion with leading term given by

$$
G_{c\pm}(x, x', y, k = \pm\mathrm{i}k_I) \approx \frac{\mathrm{e}^{-k_I^2 y-((x-x')^2/4y)}}{\sqrt{4\pi y}}\theta(y) + \mathrm{i}\sum_{|k_I|\leqslant|k_j|} g_j(x, x')\mathrm{e}^{(\kappa_j^2-k_I^2)y}\theta(y). \tag{21}
$$

Proof. On the imaginary axis, we have $A_\pm = R$ and $G_{c_\pm}$ takes the simple form

$$
G_{c_\pm}(x, x', y, k) = \frac{\theta(y)}{2\pi}\lim_{R\to\infty}\int_{-R}^{R}\mathrm{e}^{-y(l^2+2kl)}g_\pm(x, x', l+\mathrm{i}k_I)\,\mathrm{d}l\bigg|_{C'_\pm}.
$$

In view of the analyticity properties of $g_\pm(x, x', l)$ (see (9) and (10)) we can deform the relevant integral in Green's function to the real axis by considering a contour taken in the clockwise sense and with vertices at the points v_1, v_2, v_3, v_4 on the complex plane where

$$
v_1 = -R, \quad v_2 = R, \quad v_3 = R + \mathrm{i}k_I, \quad v_4 = -R + \mathrm{i}k_I.
$$

The contribution of the integrals over the vertical sides vanish as $R \to \infty$; indeed, they are proportional to

$$
\frac{1}{2\pi}\int_0^{k_I}\mathrm{e}^{-y((R+\mathrm{i}s)^2+k_I^2)}\mathrm{e}^{-\mathrm{i}(R+\mathrm{i}s)(x-x')}\,\mathrm{d}s,
$$

which tends to zero as $R \to \infty$. It follows that

$$
\begin{aligned}
G_{c\pm}(x, x', y, k = k_I) ={}& \frac{1}{2\pi}\lim_{R\to\infty}\int_{-R}^{R}\theta(y)\mathrm{e}^{-y(l^2+k_I^2)}g_\pm(x, x', l)\,\mathrm{d}l \\
&-\mathrm{i}\sum_{|k_I|>|k_j|}\mathrm{e}^{(\kappa_j^2-k_I^2)y}g_j(x, x')\theta(y),
\end{aligned} \tag{22}
$$

where the last term arises from the contribution of the poles of $g_\pm(k)$ within the contour; similarly

$$
\begin{aligned}
G_\pm(x, x', y, k = k_I) ={}& \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-y(l^2+k_I^2)}g_\pm(x, x', l)\,\mathrm{d}l - \mathrm{i}\sum_{|k_I|>|k_j|}\mathrm{e}^{(\kappa_j^2-k_I^2)y}g_j(x, x')\right] \\
&\times\theta(y) + \mathrm{i}\sum_{|k_j|\geqslant|k_I|}\mathrm{e}^{(\kappa_j^2-k_I^2)y}g_j(x, x')\theta(-y).
\end{aligned} \tag{23}
$$

The result (19) follows then by using proposition (4) corresponding to $h(l) = \mathrm{e}^{-y(l^2+k_I^2)}$.

(ii) Setting x, x' fixed, exponential decay in y follows by noting that if the potential $u_\infty(x)$ satisfies (5), then there exists a constant D such that

$$|\mathrm{e}^{\mathrm{i}l(x-x')}g_+(x,x',l)| \leqslant D|xx'|. \tag{24}$$

Dominated convergence then gives

$$\lim_{y\to\infty}\sqrt{y}\int_{-\infty}^{\infty}\mathrm{e}^{-yl^2}g_+(x,x',l)\,\mathrm{d}l = \int_{-\infty}^{\infty}\mathrm{e}^{-l^2}g_+(x,x',0)\,\mathrm{d}l.$$

Likewise, by splitting as

$$g_\pm(x,x',l) = \mathrm{e}^{-\mathrm{i}l(x-x')} + (g_\pm(x,x',l) - \mathrm{e}^{-\mathrm{i}l(x-x')})$$

and using the Riemann Lebesgue lemma, it follows that the integral in (23) vanishes as $x\to\infty$.

To pin down the precise asymptotic expansions (20) and (21) one does the following. Suppose that $x - x' > 0$ and consider

$$I \equiv \int_{-\infty}^{\infty}\mathrm{e}^{-yl^2}g_+(x,x',l)\,\mathrm{d}l.$$

Using proposition 4 and the expansion (10) one has that

$$I = \int_{-\infty}^{\infty}\mathrm{e}^{-yl^2}g_+(x,x',l)\,\mathrm{d}l = \int_{-\infty}^{\infty}\mathrm{e}^{-yl^2}g_-(x,x',l)\,\mathrm{d}l = \int_{-\infty}^{\infty}\mathrm{e}^{-yl^2-\mathrm{i}l(x-x')}\tilde{g}_-(x,x',l)\,\mathrm{d}l,$$

where $\tilde{g}_-(x,x',l)$ is meromorphic on the lower half plane. Note next that

$$\begin{aligned}\int_{-\infty}^{\infty}\mathrm{e}^{-yl^2-\mathrm{i}l(x-x')}\tilde{g}_-(x,x',l)\,\mathrm{d}l &= \lim_{R\to\infty}\int_{-R}^{R}\mathrm{e}^{-yl^2-\mathrm{i}l(x-x')}\tilde{g}_-(x,x',l)\,\mathrm{d}l\\ &= \frac{1}{\sqrt{y}}\mathrm{e}^{-t^2}\lim_{R\to\infty}\int_{-R\sqrt{y}+\mathrm{i}t}^{R\sqrt{y}+\mathrm{i}t}\mathrm{e}^{-l^2}\tilde{g}_-\left(x,x',\frac{l-\mathrm{i}t}{\sqrt{y}}\right)\mathrm{d}l\end{aligned}$$

and $t \equiv ((x-x')/2\sqrt{y})$. Now, consider the integral

$$\int_{\Gamma_R}\mathrm{e}^{-l^2}\tilde{g}_-\left(x,x',\frac{l-\mathrm{i}t}{\sqrt{y}}\right)\mathrm{d}l,$$

where Γ_R is a rectangular contour taken in the positive sense and with vertices at the following points on the complex plane:

$$v_1 = -R\sqrt{y},\quad v_2 = R\sqrt{y},\quad v_3 = R\sqrt{y}+\mathrm{i}t,\quad v_4 = -R\sqrt{y}+\mathrm{i}t.$$

The integrals over the vertical sides vanish as $R\to\infty$; indeed,

$$\begin{aligned}\lim_{R\to\infty}\int_{\overline{v_2v_3}}\mathrm{e}^{-l^2}\tilde{g}_-\left(x,x',\frac{l-\mathrm{i}t}{\sqrt{y}}\right) &= \mathrm{i}\lim_{R\to\infty}\int_0^t\mathrm{e}^{-(\mathrm{i}s+R\sqrt{y})^2}\tilde{g}_-\left(x,x',\frac{R+\mathrm{i}s-\mathrm{i}t}{\sqrt{y}}\right)\mathrm{d}s\\ &= \mathrm{i}\int_0^t\lim_{R\to\infty}\mathrm{e}^{-(\mathrm{i}s+R\sqrt{y})^2}\tilde{g}_-\left(x,x',\frac{R+\mathrm{i}s-\mathrm{i}t}{\sqrt{y}}\right)\mathrm{d}s = 0,\end{aligned}$$

as the limit can be taken under the integral sign.

Next, notice that

$$\mathrm{e}^{-l^2}\tilde{g}_-\left(x,x',\frac{l-\mathrm{i}t}{\sqrt{y}}\right),$$

exists and is a meromorphic function for $l_I \leqslant t$; whenever $t \geqslant \kappa_j\sqrt{y}$, it has poles at the points $l_j \equiv \mathrm{i}(t-\kappa_j\sqrt{y})$ with residues

$$\mathrm{Res}_{l=l_j}\mathrm{e}^{-l^2}\tilde{g}_-\left(x,x',\frac{l-\mathrm{i}t}{\sqrt{y}}\right) = -\sqrt{y}g_j(x,x')\mathrm{e}^{(x^2/4y)+\kappa_j^2y}.$$

It follows that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-yl^2-\mathrm{i}l(x-x')}\tilde{g}_-(x,x',l)\,\mathrm{d}l = \frac{\mathrm{e}^{-t^2}}{\sqrt{y}}\int_{-\infty}^{\infty}\mathrm{e}^{-l^2}\tilde{g}_-\left(x,x',\frac{l-\mathrm{i}t}{\sqrt{y}}\right)\mathrm{d}l$$
$$+2\pi\mathrm{i}\sum_{t\geqslant\kappa_j\sqrt{y}} g_j(x,x')\mathrm{e}^{\kappa_j^2 y}.$$

If $|x-x'|/y\to\infty$ then $((l-\mathrm{i}t)/\sqrt{y})\to -\mathrm{i}\infty$; $\tilde{g}_-(x,x',((l-\mathrm{i}t)/\sqrt{y}))\to 1$. □

If $y\to\infty$ with $(x-x')/(2y)\equiv l_0$ fixed, then, $((l-\mathrm{i}t))/\sqrt{y}\to -\mathrm{i}l_0$; $\tilde{g}_-(x,x',((l-\mathrm{i}t)/\sqrt{y}))\to\tilde{g}_-(x,x',-\mathrm{i}l_0)$. The result is obtained on integration.

Proposition 7. *On the real axis, $k_I=0$, Green's function satisfies the following conditions.*

(i) It is an even function of k_R:

$$G_\pm(x,x',y,k_R) = G_\pm(x,x',y,-k_R). \tag{25}$$

(ii) It is continuous across the line $k_I=0$:

$$G_+(x,x',y,k_R) = G_-(x,x',y,k_R), \qquad k_R\in R. \tag{26}$$

Proof. We only need to prove the latter properties for the continuous part as they are clear for the discrete one. Let $k_I=0$, $k=k_R$. Then,

$$G_{c\pm}(x,x',y,k) = \frac{1}{2\pi}\left[\int_{(-\infty,-|k_R|)\cup(|k_R|,\infty)}\theta(y) - \int_{-|k_R|}^{|k_R|}\theta(-y)\right]\mathrm{e}^{-y(l^2-k_R^2)}g_\pm(x,x',l)\,\mathrm{d}l$$
$$\equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} h(k_R,y,l)g_\pm(x,x',l)\,\mathrm{d}l,$$

where

$$\mathrm{e}^{y(l^2-k_R^2)}h(k_R,y,l) = \chi_{(-\infty,-|k_R|]\cup(|k_R|,\infty)}(l)\theta(y) - \chi_{(-|k_R|,|k_R|]}(l)\theta(-y).$$

Equation (25) then follows immediately, while (26) follows from proposition (4). □

The asymptotic behavior of Green's function on the imaginary axis has been already determined (proposition 6). We consider next a similar result away from the imaginary axis.

Proposition 8.

(i) As $x^2+y^2\to\infty$ with x' fixed and $k_R\neq 0$ Green's function has an asymptotic expansion with leading term

$$G_\pm(x,x',y,k) = \frac{-\lambda\mathrm{e}^{\mathrm{i}qy}g_\pm(x,x',-\bar{k})}{2\bar{k}y-\mathrm{i}(x-x')} - \frac{\lambda g_\pm(x,x',k)}{2ky+\mathrm{i}(x-x')} + O\left(\frac{1}{x^2+y^2}\right), \tag{27}$$

where

$$\lambda\equiv -\frac{\mathrm{sign}k_R}{2\pi}; \qquad p=-2k_R, \quad q\equiv 4k_Rk_I, \tag{28}$$

or with

$$\tilde{G}_\pm(x,x',y,k)\equiv \mathrm{e}^{\mathrm{i}k(x-x')}G_\pm(x,x',y,k); \quad \tilde{g}_\pm(x,x',k)\equiv \mathrm{e}^{\mathrm{i}k(x-x')}g_\pm(x,x',k), \tag{29}$$

$$\tilde{G}_\pm(x,x',y,k) = \frac{-\lambda\mathrm{e}^{\mathrm{i}qy+2\mathrm{i}k_R(x-x')}\tilde{g}_\pm(x,x',-\bar{k})}{2\bar{k}y-\mathrm{i}(x-x')} - \frac{\lambda\tilde{g}_\pm(x,x',k)}{2ky+\mathrm{i}(x-x')} + O\left(\frac{1}{x^2+y^2}\right)$$

(ii) As $|k| \to \infty$ with $y \neq 0$ one has the following.

If $|k_I| \to \infty$, keeping k_R constant and with $y \neq 0$, Green's function has an asymptotic expansion with leading term

$$G_\pm(x, x', y, k) = \lambda \frac{\mathrm{e}^{\mathrm{i}qy} g_\pm(x, x', -\bar{k}) - g_\pm(x, x', k)}{2\mathrm{i}k_I y} + O\left(\frac{1}{k_I^2}\right). \tag{30}$$

As $|k_R| \to \infty$, keeping k_I constant and $y \neq 0$, Green's function has an asymptotic expansion with leading term

$$G_\pm(x, x', y, k) = -\frac{\lambda\theta(-y)}{2yk}(g_\pm(x, x', k) + \mathrm{e}^{\mathrm{i}qy} g_\pm(x, x', -\bar{k})) + O\left(\frac{1}{k_R^2}\right). \tag{31}$$

We next sketch a proof of the above facts.

(i) Note that one has

$$\begin{aligned} G_{c\pm}(x, x', y, k) &= \frac{1}{2\pi}\left[\int_{(-\infty,-|k_R|]\cup(|k_R|,\infty)} \theta(y) - \int_{-|k_R|}^{|k_R|} \theta(-y)\right] \\ &\quad \times \mathrm{e}^{-y(l^2-k_R^2+2\mathrm{i}k_I(l-k_R))} g_\pm(x, x', l + \mathrm{i}k_I)\,\mathrm{d}l \\ &\equiv \frac{1}{2\pi}\left[\int_{(-\infty,-|k_R|]\cup(|k_R|,\infty)} \theta(y) - \int_{-|k_R|}^{|k_R|} \theta(-y)\right] \\ &\quad \times \mathrm{e}^{-y(l^2-k_R^2+2\mathrm{i}k_I(l-k_R))-\mathrm{i}(l+\mathrm{i}k_I)(x-x')} \tilde{g}_\pm(x, x', l + \mathrm{i}k_I)\,\mathrm{d}l, \end{aligned}$$

where $\tilde{g}_\pm(x, x', k)$ is bounded. If $k_R = 0$ there are no endpoints and the integral is exponentially decreasing (cf [20, 21]). Otherwise, the dominant contribution comes from the endpoints and can be evaluated using integration by parts or the steepest descent method (see [43] in this regard).

(ii) If $k_R \to \infty$ and k_I is left fixed, or if $k_I \to \infty$ and k_R is left fixed the result can be established by using integration by parts again or the steepest descent method.

Remark. The asymptotic expansions (20), (21) and (27) require, to match as $k_R \to 0$, that the natural definition $\lambda(k_R = 0) = 0$ be taken.

Analyticity properties in the k-plane

Recall that whenever the discrete spectrum of the associated SO is not empty: $\{k_j : a(k_j) = 0\}_{j=1,\ldots,N} \neq \emptyset$, the integrand in Green's function has pole singularities at the points $l = -k_R$ when $k = \pm\mathrm{i}\kappa_j$. These singularities are reflected in the behaviour of G. One has the following proposition.

Proposition 9. *The continuous part $G_{c\pm}(x, x', y, k)$ satisfies the following properties.*

(i) It is finite while $\tilde{G}_{c\pm}(x, x', y, k)$ is bounded ($\tilde{G}$ is defined in (29)).

(ii) Away from the set $\{k_j\}_{j=1,\ldots,N}$ it is continuous. If $k_R \neq 0$ it has bounded jump discontinuities across the lines $k_I = \kappa_j$ with a jump

$$G_{c\pm}(k_R + \mathrm{i}\kappa_j^+) - G_{c\pm}(k_R + \mathrm{i}\kappa_j^-) = \pm\mathrm{i}\mathrm{e}^{y(k_R^2 \pm 2\mathrm{i}k_R\kappa_j))} g_j(x, x')\theta(-y). \tag{32}$$

If $k_R = 0$ it has bounded discontinuities at the points $k_I = \pm\kappa_j$ where the directional limits exist and with a jump

$$G_{c\pm}(\mathrm{i}\kappa_j^+) - G_{c\pm}(\mathrm{i}\kappa_j^-) = \mp\mathrm{i}g_j(x, x')\theta(y). \tag{33}$$

(iii) One has that

$$\lim_{k_R\to 0}\left(\lim_{k_I\to\kappa_j^+}-\lim_{k_I\to\kappa_j^-}\right)G_c(k_R+\mathrm{i}k_I)\neq\left(\lim_{k_I\to\kappa_j^+}-\lim_{k_I\to\kappa_j^-}\right)G_c(\mathrm{i}k_I), \tag{34}$$

but

$$\lim_{k_R\to 0}G_c(k_R+\mathrm{i}k_I)=G_c(\mathrm{i}k_I), \qquad \forall k_I\neq\kappa_j \tag{35}$$

and

$$\left(\lim_{k_I\to\kappa_j^+}-\lim_{k_I\to\kappa_j^-}\right)\lim_{k_R\to 0}G_c(k_R+\mathrm{i}k_I)=\left(\lim_{k_I\to\kappa_j^+}-\lim_{k_I\to\kappa_j^-}\right)G_c(\mathrm{i}k_I). \tag{36}$$

Proof. We set

$$G_{c\pm}(x,x',y,k)=\frac{1}{2\pi}\left[\int_{(-\infty,-|k_R|]\cup(|k_R|,\infty)}\theta(y)-\int_{-|k_R|}^{|k_R|}\theta(-y)\right]\frac{h_\pm(l+\mathrm{i}k_I)}{l+\mathrm{i}(k_I\mp\kappa_j)}\,\mathrm{d}l,$$

where we define $z_I\equiv k_I\mp\kappa_j$,

$$h_\pm(x,x',y,l+\mathrm{i}z_I)=\mathrm{e}^{-y([l+\mathrm{i}k_I]^2-k^2)}(l+\mathrm{i}z_I)g_\pm(x,x',l+\mathrm{i}z_I).$$

Hence, we have that $G_{c\pm}(x,x',y,k)\equiv R_\pm(x,x',y,k)+S_\pm(x,x',y,k)$, where, if $k_R\neq 0$,

$$2\pi R_\pm=\int_{(-\infty,-|k_R|]\cup(|k_R|,\infty)}\theta(y)\frac{h_\pm(l+\mathrm{i}k_I)}{l+\mathrm{i}z_I}\,\mathrm{d}l-\int_{-|k_R|}^{|k_R|}\theta(-y)\frac{h_\pm(l+\mathrm{i}k_I)-h_\pm(\pm\mathrm{i}\kappa_j)}{l+\mathrm{i}z_I}\,\mathrm{d}l.$$

Note that

$$\frac{h_\pm(l+\mathrm{i}k_I)-h_\pm(\pm\mathrm{i}\kappa_j)}{l+\mathrm{i}z_I}\leqslant\sup\frac{\partial}{\partial l}h_\pm(l)$$

and hence R is regular at the singularity $k_I=\kappa_j$. $S_\pm$ contains the singular part of $G_{c\pm}$ at the singularity and is given by

$$S_\pm=-\frac{1}{2\pi}\int_{-|k_R|}^{|k_R|}\theta(-y)\frac{h_\pm(\pm\mathrm{i}\kappa_j)}{l+\mathrm{i}z_I}\,\mathrm{d}l=\mp\mathrm{e}^{y(k_R^2+2\mathrm{i}k_Rk_I+\kappa_j^2-k_I^2))}g_j(x,x')\theta(-y)F(k_R,k_I),$$

where F is the function

$$F(k_R,k_I)\equiv\frac{1}{2\pi}\log\frac{|k_R|+\mathrm{i}z_I}{-|k_R|+\mathrm{i}z_I}.$$

This function is always bounded and has a discontinuity across $k_I=\kappa_j$ with a jump

$$F(k_R+\mathrm{i}\kappa_j^+)-F(k_R+\mathrm{i}\kappa_j^-)=-\mathrm{i}.$$

Then, equation (32) follows.

If $k_R=0$, the regular and the singular parts are

$$R_\pm=\frac{\theta(y)}{2\pi}\left[\int_{(-\infty,-1]\cup(|1,\infty)}\frac{h_\pm(l+\mathrm{i}z_I)}{l+\mathrm{i}z_I}\,\mathrm{d}l+\int_{(-1,1)}\frac{h_\pm(l+\mathrm{i}z_I)-h_\pm(\pm\mathrm{i}\kappa_j)}{l+\mathrm{i}z_I}\,\mathrm{d}l\right],$$

$$S_\pm=\frac{1}{2\pi}\int_{(-1,1)}\theta(y)\frac{h_\pm(\pm\mathrm{i}\kappa_j)}{l+\mathrm{i}z_I}\,\mathrm{d}l=\pm\theta(y)g_j(x,x')F(1,k_I),$$

where $F(1,k_I)$ is bounded and has a discontinuity across $k_I=\pm\kappa_j$ with a jump $-\mathrm{i}$. This gives (33). From (32) and (33), (34) follows.

(iii) To study the limit $\lim_{k_R\to 0}$ of Green's function we note that if $k_R \neq 0$ one can split further as

$$\int_{(-\infty,-|k_R|]\cup(|k_R|,\infty)} \frac{h_\pm(l+\mathrm{i}z_I)}{l+\mathrm{i}z_I}\,\mathrm{d}l = r_1+r_2+r_3,$$

where

$$r_1 = \int_{(-\infty,-1]\cup(1,\infty)} \frac{h_\pm(l+\mathrm{i}z_I)}{l+\mathrm{i}z_I}\,\mathrm{d}l,$$

$$r_2 = \int_{(-1,-|k_R|]\cup(|k_R|,1]} \frac{h_\pm(l+\mathrm{i}z_I)-h_\pm(\pm\mathrm{i}\kappa_j)}{l+\mathrm{i}z_I}\,\mathrm{d}l,$$

$$r_3 = \int_{(-1,-|k_R|]\cup(|k_R|,1]} \frac{h_\pm(\pm\mathrm{i}\kappa_j)\mathrm{d}l}{l+\mathrm{i}z_I} = \pm g_j(x,x')\log\frac{1+\mathrm{i}z_I}{-1+\mathrm{i}z_I}\,\frac{-|k_R|+\mathrm{i}z_I}{|k_R|+\mathrm{i}z_I}$$

and hence

$$G_{c\pm}(x,x',y,k) = \frac{\theta(y)}{2\pi}[r_1+r_2+r_3] \mp \frac{\theta(-y)}{2\pi}\Big[-\mathrm{e}^{y(k_R^2+2\mathrm{i}k_Rk_I+\kappa_j^2-k_I^2))}g_j(x,x')F(k_R,k_I)$$
$$+\int_{-|k_R|}^{|k_R|}\frac{h_\pm(l+\mathrm{i}k_I)-h_\pm(\pm\mathrm{i}\kappa_j)}{l+\mathrm{i}z_I}\,\mathrm{d}l\Big].$$

If $k_R = 0$, we set

$$\int_{(-\infty,\infty)} \frac{h_\pm(l+\mathrm{i}k_I)}{l+\mathrm{i}z_I}\,\mathrm{d}l = r_1'+r_2'+r_3'$$

with $r_1' = r_1$

$$r_2' = \int_{(-1,1)} \frac{h_\pm(l+\mathrm{i}z_I)-h_\pm(\pm\mathrm{i}\kappa_j)}{l+\mathrm{i}z_I}\,\mathrm{d}l,$$

$$r_3' = \int_{(-1,1)} \frac{h_\pm(\pm\mathrm{i}\kappa_j)\,\mathrm{d}l}{l+\mathrm{i}z_I} = \pm\frac{F(1,k_I)}{2\pi}g_j(x,x').$$

As $k_R \to 0$, one has

$$\lim_{k_R\to 0} F(k_R,k_I) = 0; \qquad \lim_{k_R\to 0}\int_{-|k_R|}^{|k_R|}\frac{h_\pm(l+\mathrm{i}k_I)-h_\pm(\pm\mathrm{i}\kappa_j)}{l+\mathrm{i}z_I}\,\mathrm{d}l = 0,$$

$$\lim_{k_R\to 0} = r_3'; \qquad \lim_{k_R\to 0} r_2 = r_2'.$$

This implies (35) and (36). □

The study of the analyticity properties of the discrete part is immediate and left for the reader. Adding both contributions we obtain the following proposition.

Proposition 10. *Green's function $G_\pm(x,x',y,k)$ satisfies these important properties.*

- *It is finite while $\tilde{G}_\pm(x,x',y,k)$ is bounded.*
- *It is continuous everywhere except at the points $k_R = 0$, $k_I = \kappa_j$. In particular,*

$$G(k_R+\mathrm{i}\kappa_j^+) - G(k_R+\mathrm{i}\kappa_j^-) = 0, \qquad k_R \neq 0, \tag{37}$$

$$G(\mathrm{i}\kappa_j^+) - G(\mathrm{i}\kappa_j^-) = \mp\mathrm{i}g_j(x,x') \tag{38}$$

and G is continuous if there is no discrete spectrum.

We next evaluate the departure from holomorphicity of Green's function via its $\bar{\partial}$ derivative. Motivated by the fact that the $\bar{\partial}$ derivative of Green's function is discontinuous on the imaginary axis (even for the decaying problem of [5]), we take the natural definition

$$2\frac{\partial f}{\partial \bar{k}} \equiv \left(\frac{\partial f}{\partial k_R} + \mathrm{i}\frac{\partial f}{\partial k_I}\right)(k_R^+, k_I) + \left(\frac{\partial f}{\partial k_R} + \mathrm{i}\frac{\partial f}{\partial k_I}\right)(k_R^-, k_I).$$

Proposition 11. *The departure from holomorphicity of Green's function is given in terms of the quantities (28) by the following:*

(i) If $k_R \neq 0$

$$\frac{\partial G_\pm}{\partial \bar{k}}(x, x', y, k) = \lambda \mathrm{e}^{\mathrm{i}qy} g_\pm(x, x', -\bar{k}). \tag{39.1}$$

(ii) If $k_R = 0$, *then,*

$$\frac{\partial G_\pm}{\partial \bar{k}}(x, x', y, k = \mathrm{i}k_I) = \pm\frac{1}{2}\sum_j g_j(x, x')\delta(k_I \mp \kappa_j). \tag{39.2}$$

Remark. Equation (39.1) indicates that the DBAR derivative of G has a jump at $k_R = 0$. The symmetric definition of $\bar{\partial}$ amounts to defining $\lambda(k_R = 0) = 0$. Note that the use of other definitions of $\bar{\partial}$ results in the addition of a term proportional to $g_\pm(x, x', k_I)$ on the right-hand side of (39.2). The discontinuity at $k_R = 0$ of the $\bar{\partial}$ derivative is already present for the decaying problem (cf [5]); we also note that these jump discontinuities are irrelevant since they only affect the IP on a set of zero measure. The novelties for the nondecaying problem are: (i) the pole present in (39.1), which yields a singularity in the IP; and, (ii) the appearance of the $\delta(k_I \mp \kappa_j)$ term in the right-hand side of (39.2), even when the symmetric choice of $\bar{\partial}$, alluded to above, is taken. However, there is still *no contribution* to the IP arising from the latter term since the IP involves a double integral (i.e. all contributions to the IP come from (39.1)).

Proof. By direct calculation on formula (14) and use of the relation $(\partial/\partial\bar{k})(1/(k - k_j)) = \pi\delta(k - k_j)$ we find

$$\frac{\partial G_{c\pm}}{\partial \bar{k}}(x, x', y, k) = \lambda \mathrm{e}^{\mathrm{i}qy} g_\pm(x, x', -\bar{k}) \pm \frac{1}{2}\sum_j \mathrm{e}^{y(k_R^2 \pm 2\mathrm{i}k_R\kappa_j)} g_j(x, x').$$

$$\left(\int_{(-\infty,-|k_R|]\cup(|k_R|,\infty)} \theta(y) - \int_{-|k_R|}^{|k_R|)} \theta(-y)\right)\delta(l)\delta(k_I \mp \kappa_j)\,\mathrm{d}l.$$

If $k_R \neq 0$ this gives

$$\frac{\partial G_{c\pm}}{\partial \bar{k}} = \lambda \mathrm{e}^{\mathrm{i}qy} g_\pm(x, x', -\bar{k}) \mp \frac{1}{2}\sum_j \mathrm{e}^{y(k_R^2 \pm 2\mathrm{i}k_R\kappa_j)} g_j(x, x')\delta(k_I \mp \kappa_j)\theta(-y)\int_{-|k_R|}^{|k_R|}\delta(l)\,\mathrm{d}l.$$

If $k_R = 0$, taking into account the symmetric definition for the $\bar{\partial}$ derivative, we obtain

$$\frac{\partial G_{c\pm}}{\partial \bar{k}}(x, x', y, k) = \pm\frac{1}{2}\sum_j g_j(x, x')\delta(k_I \mp \kappa_j)\theta(y)\int_{-\infty}^{\infty}\delta(l)\,\mathrm{d}l.$$

Likewise using $\partial/\partial\bar{k} = \frac{1}{2}(\partial/\partial k_R + \mathrm{i}\partial/\partial k_I)$ in the discrete part of Green's function we find that

$$\frac{\partial G_{d\pm}}{\partial \bar{k}}(x, x', y, k) = \pm\frac{1}{2}\sum_j \delta(k_I \mp \kappa_j)\mathrm{e}^{(k^2+\kappa_j^2)y} g_j(x, x')\theta(-y).$$

Using the fact that $\int_{-|k_R|}^{|k_R|} \delta(l)\,\mathrm{d}l = 1,\ \forall k_R \neq 0$ and adding the contributions from the discrete and continuous part we recover (39) upon cancellation of similar terms for $k_R \neq 0$. □

Proposition 12. *Green's function satisfies the symmetry property*

$$G(x, x', y, k) = \mathrm{e}^{\mathrm{i}qy} G(x, x', y, -\bar{k}). \tag{40}$$

The proof is exactly the same as for the decaying case and is omitted.

6. The direct and inverse problems

Recall that we consider a solution $\nu(x, y, k)$ to (3), defined for all values of the spectral parameter $k \in C$, and which satisfies $\lim_{x^2+y^2\to\infty} \nu(x, y, k)\mathrm{e}^{\mathrm{i}kx} = 1$.

Such a function solves the integral equation

$$\nu(x, y, k) = h(x, k) + \int \mathrm{d}x'\,\mathrm{d}y' G(x, x', y - y', k) U \nu(x', y', k), \tag{41}$$

where

$$\nu = \nu_+(x, k)\theta(k_I) + \nu_-(x, k)\theta(-k_I); \qquad h(x, k) = \frac{\phi_+}{a(x, k)}\theta(k_I) + \psi_-(x, k)\theta(-k_I). \tag{42}$$

Here, both $G(x, x', y, k)$ and $h(x, k)$ have different representations on $C_\pm$ (the upper/lower half k planes).

In the preceding discussion we have proved that $\tilde{G}(x, x', y, k)$ is bounded and is continuous everywhere except at the points $k = \pm\mathrm{i}\kappa_j$, where it is bounded and the directional limits exist. It follows from (27), (30) and (31) that it is decaying as either $|x|$, $|y|$ or $|k|$ go to infinity, and we have detailed the decay rate; define the bounded function $\phi_+(x, k) \equiv \tilde{\phi}_+(x, k)\mathrm{e}^{-\mathrm{i}kx}$. From these considerations, we can formulate the following theorem.

Theorem 13. *Suppose that the potential $u_\infty(x)$ satisfies equation (5) and that $U(x, y)$ is in $L_\infty \cap L_1((1 + x^2)\,\mathrm{d}x\,\mathrm{d}y)$ having a suitable small norm:*

$$|U(x, y)| \leqslant C; \qquad \int (1 + x^2)|U(x, y)|\,\mathrm{d}x\,\mathrm{d}y < \epsilon \tag{43}$$

for some suitable constants C and ϵ with ϵ small enough. Let

$$h_1(x, k) = \mathrm{e}^{\mathrm{i}kx}(\phi_+(x, k)\theta(k_I) + \psi_-(x, k)\theta(-k_I)).$$

Then, the integral equation

$$\mu_1(x, y, k) = h_1(x, k) + \int \mathrm{d}x'\,\mathrm{d}y' \tilde{G}(x, x', y - y', k) U \mu_1(x', y', k) \tag{44}$$

has a unique solution $\mu_1(x, y, k)$ on $C - \{k_j\}_{j=1,\ldots,N}$; it is bounded and continuous in the k plane and it may increase linearly with $|x|$. Besides

$$\mu(x, y, k) = \mu_1(x, y, k)\left(\frac{\theta(k_I)}{a(k)} + \theta(-k_I)\right)$$

exists, is bounded and continuous away from the denumerable set $\{k_j\}_{j=1,\ldots,N}$, while $\nu(x, y, k) \equiv \mu(x, y, k)\mathrm{e}^{-\mathrm{i}kx}$ exists, solves (41), is finite and continuous away from the set $\{k_j\}_{j=1,\ldots,N}$. Generically, these functions are not holomorphic anywhere.

Proof. A more complete account of rigorous properties and norm estimates of eigenfunctions of the spectral operator $\hat{L}$ will be the subject of a future publication. Here, we note the following. Using (10), we see that the integral equation (44) has bounded forcing and a Green's function that satisfies a suitable bound away from the eigenvalues. An application of the principle of contraction mappings yields the result that $\mu_1(x, y, k)$ exists, is bounded and continuous there. □

Proposition 14. *Suppose that the potentials $u_\infty(x)$ and $U(x, y)$ satisfy the conditions of theorem 13. The following properties hold.*

(i) $\mu_+(x, y, k)$ can be split as

$$\mu_+(x, y, k) = \frac{e^{ikx}}{a(k)}(\phi_+(x, k) + e(x, y, k)), \tag{45}$$

where the function $e(x, y, k)$ satisfies the following conditions.

(i.1) If $k_R \neq 0$ the function $e(x, y, k)$ has an asymptotic expansion as $x^2 + y^2 \to \infty$ with leading term

$$e(x, y, k) = -A(k)\frac{\phi_+(x, k)}{2ky + ix} - \phi_+(x, -\bar{k})\frac{F_+(k)a(k)}{a(-\bar{k})}\frac{e^{iqy}}{2\bar{k}y - ix} + O\left(\frac{1}{x^2 + y^2}\right), \tag{46}$$

where $q = 4k_R k_I$ (see (28)) and we define the bidimensional scattering data of the problem as

$$F_+(k) \equiv \lambda \int dx\, dy\, e^{-iqy} U \nu_+(x, y, k)\psi_+(x, -\bar{k}),$$

$$F_-(k) \equiv -\frac{\lambda}{a_-(-\bar{k})}\int dx\, dy\, e^{-iqy} U \nu_-(x, y, k)\phi_-(x, -\bar{k}),$$

$$A(k) \equiv \lambda \int dx\, dy U \nu_+(x, y, k)\psi_+(x, k). \tag{47}$$

(i.2) If $k_R = 0$ then

$$\lim_{|x|\to\infty} e(x, y, ik_I) = 0, \qquad \forall k_I, \tag{48.1}$$

$$\lim_{|y|\to\infty} e(x, y, ik_I) = 0, \qquad \forall k_I \neq |k_j| \tag{48.2}$$

(ii) As $|k_R| \to \infty$, it has the asymptotic expansion

$$\mu(x, y, k) = \left(1 + \sum_{n=1}^{\infty} \mu^{n)}(x, x')/k^n)\right), \tag{49}$$

where the coefficients are uniformly bounded. The first few terms are

$$\mu(k) = 1 + \frac{\mu^{1)}(x, y)}{k} + O\left(\frac{1}{k^2}\right), |k| \to \infty \qquad \text{where } 2ik\partial_x \mu^{1)} = -u. \tag{50}$$

Proof. This expansion (45) for the eigenfunction follows upon insertion of the asymptotic expansions of Green's function (see propositions 6 and 8) into the integral equation (44) and estimation of the relevant terms. We skip the details. Equation (50) is obtained using the expansions (30) and (31) in the integral equation (44) and the fact that theorem 13 guarantees the existence of a bounded solution. □

The IP involves reconstructing the eigenfunction $\nu(k)$ from appropriate data that define its departure from holomorphicity. One has the following theorem.

Theorem 15.

(i) The departure from holomorphicity of the function $\mu(x, y, k) \equiv \nu(x, y, k)\mathrm{e}^{\mathrm{i}kx}$ *satisfies*

$$\frac{\partial \mu}{\partial \bar{k}} = \pi \sum_{1}^{N} C_j \mathrm{e}^{-2\kappa_j x} \mu_-(-\mathrm{i}\kappa_j)\delta(k_R)\delta(k_I - \kappa_j) + \mathrm{e}^{\mathrm{i}[qy+(k+\bar{k})x]} \sum_{\pm} F_\pm(k)\mu_\pm(-\bar{k})\theta(\pm k_I)$$
$$- \frac{\rho(k)}{2\mathrm{i}} \mathrm{e}^{2\mathrm{i}kx} \mu_-(-k)\delta(k_I), \tag{51}$$

where $q = 4k_R k_I$ *(see (28)).*

(ii) $\mu(x, y, k)$ *satisfies the equation of the IP*

$$\mu(x, y, k) = 1 + \sum_{j=1}^{N} C_j \mathrm{e}^{-2\kappa_j x} \frac{\mu_-(-\mathrm{i}\kappa_j)}{k - \mathrm{i}\kappa_j} + \frac{1}{2\pi\mathrm{i}} \sum_{\pm} \int_{C_\pm} F_\pm(z)\mu_\pm(-\bar{z}) \frac{\mathrm{e}^{\mathrm{i}[qy+2z_R x]}}{z-k}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}$$
$$+ \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}z_R \frac{\rho(z_R)\mathrm{e}^{2\mathrm{i}zx}\mu_-(-z_R)}{z_R - k}. \tag{52}$$

(iii) The potential is given in terms of scattering data by

$$u = \frac{\partial}{\partial x}\Bigg[2\mathrm{i} \sum_{1}^{N} C_j \mathrm{e}^{-2\kappa_j x} \mu_-(-\mathrm{i}\kappa_j) - \frac{1}{\pi} \sum_{\pm} \int_{C_\pm} F_\pm(z)\mu_\pm(-\bar{z}) \mathrm{e}^{\mathrm{i}[qy+2z_R x]}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}$$
$$- \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}z_R \rho(z_R) \mathrm{e}^{2\mathrm{i}z_R x} \mu_-(-z_R)\Bigg]. \tag{53}$$

Proof. Note first that the jump of $\nu(x, y, k)$ on the real axis is given by

$$[\nu_+ - \nu_-](x, y, k) = \rho(k)\nu_-(x, y, -k). \tag{54}$$

To prove (54), note that in view of (6) and the fact that Green's function is continuous across the real axis (see (26)) the only jump in $\nu(x, y, k)$ on the real axis is due to $h(x, k)$ in (42). It follows that $\Delta(x, y, k) \equiv \nu_+(x, y, k) - \nu_-(x, y, k)$ satisfies the integral equation

$$\Delta(x, y, k) = \rho(k)\psi_+(x, k) + \int \mathrm{d}x'\, \mathrm{d}y'\, UG_-(x, x', y, k)\Delta(x', y', k).$$

From (25) we obtain that it also solves

$$\Delta(x, y, k) = \rho(k)\psi_-(x, -k) + \int \mathrm{d}x'\, \mathrm{d}y'\, UG_-(x, x', y, -k)\Delta(x', y', k)$$

and (54) follows.

Operating with $\partial/\partial\bar{k}$ on equation (41) and using (39) we find the following integral equation for $\partial\nu_\pm/\partial\bar{k}$ corresponding to $k \in C_\pm$:

$$\frac{\partial \nu_\pm}{\partial \bar{k}}(k) = \pi \sum_{j=1}^{N} C_j \psi_j(x)\delta(k_R)\delta(k_I - \kappa_j) + \mathrm{e}^{\mathrm{i}qy} F_\pm(k) h_\pm(x, -\bar{k})$$
$$+ \int \mathrm{d}x'\, \mathrm{d}y' G_\pm(x, x', y - y', k) U \frac{\partial \nu_\pm(k)}{\partial \bar{k}}. \tag{55}$$

Let the function $\eta(x, y, k)$ satisfy

$$\eta(x, y, k) = \mathrm{e}^{\mathrm{i}qy} h_\pm(x, -\bar{k}) + \int \mathrm{d}x'\, \mathrm{d}y'\, UG_\pm(x, x', y, k)\eta(x', y').$$

Using (40) one has that $\hat{\eta} \equiv \mathrm{e}^{-\mathrm{i}qy}\eta$ solves

$$\hat{\eta}(x, y, k) = h_{\pm}(x, -\bar{k}) + \int \mathrm{d}x'\,\mathrm{d}y'\, UG_{\pm}(x, x', y, -\bar{k})\hat{\eta}(x', y').$$

Thus,

$$\hat{\eta}(x, y, k) = \nu(x, y, -\bar{k}).$$

Likewise let the function $\delta_j(x, y)$ solve at $k = \mathrm{i}\kappa_j$

$$\begin{aligned}\delta_j(x, y) &= \psi_j(x) + \int \mathrm{d}x'\,\mathrm{d}y'\, UG_{+}(x, x', y, \mathrm{i}\kappa_j)\delta_j(x', y') \\ &= \psi_{+}(x, \mathrm{i}\kappa_j) + \int \mathrm{d}x'\,\mathrm{d}y'\, UG_{+}(x, x', y, \mathrm{i}\kappa_j)\delta_j(x', y')\end{aligned}$$

or, using (19),

$$\delta_j(x, y) = \psi_{-}(x, -\mathrm{i}\kappa_j) + \int \mathrm{d}x'\,\mathrm{d}y' UG_{-}(x, x', y, -\mathrm{i}\kappa_j)\delta_j(x', y'),$$

whereupon it follows that

$$\delta_j(x, y) = \nu_{-}(x, y, -\mathrm{i}\kappa_j).$$

Hence, from (55) and the latter results we have

$$\frac{\partial \nu}{\partial \bar{k}} = \pi \sum_{1}^{N} C_j \nu_{-}(-\mathrm{i}\kappa_j)\delta(k_R)\delta(k_I - \kappa_j) + \mathrm{e}^{\mathrm{i}qy} \sum_{\pm} F_{\pm}(k)\nu_{\pm}(-\bar{k})\theta(\pm k_I) - \frac{\rho(k)}{2\mathrm{i}}\nu_{-}(-k)\delta(k_I) \tag{56}$$

and

$$\begin{aligned}\frac{\partial \mu}{\partial \bar{k}} = \pi \sum_{1}^{N} C_j \mathrm{e}^{-2\kappa_j x}\mu_{-}(-\mathrm{i}\kappa_j)\delta(k_R)\delta(k_I - \kappa_j) + \mathrm{e}^{\mathrm{i}[qy+(k+\bar{k})x]} \sum_{\pm} F_{\pm}(k)\mu_{\pm}(-\bar{k})\theta(\pm k_I) \\ -\frac{\rho(k)}{2\mathrm{i}}\mathrm{e}^{2\mathrm{i}kx}\mu_{-}(-k)\delta(k_I).\end{aligned} \tag{57}$$

Once the departure from holomorphicity of the function μ is evaluated, the IP is formulated using the generalized Cauchy formula

$$\mu(k) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_\infty} \frac{\mu(z)}{z - k}\,\mathrm{d}z + \frac{1}{2\pi\mathrm{i}} \int_{C} \frac{\partial \mu/\partial \bar{z}}{z - k}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}, \tag{58}$$

where C is the complex plane and Γ_∞ its boundary. In terms of the scattering data described above, (52) is obtained.

(iii) Use that the IP is formulated using the normalization (49) and (50). □

Remark.

1. From the equations of the IP one obtains, in particular, that

$$\begin{aligned}\mu_{-}(x, y, k) = 1 + \sum_{j=1}^{N} C_j \mathrm{e}^{-2\kappa_j x}\frac{\mu_{-}(-\mathrm{i}\kappa_j)}{k - \mathrm{i}\kappa_j} + \frac{1}{2\pi\mathrm{i}} \sum_{\pm} \int_{C_\pm} F_{\pm}(z)\mu_{\pm}(-\bar{z})\frac{\mathrm{e}^{\mathrm{i}[qy+2z_R x]}}{z - k + \mathrm{i}\epsilon}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \\ + \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}z \frac{\rho(z)\mathrm{e}^{2\mathrm{i}zx}\mu_{-}(-z)}{z - k + \mathrm{i}\epsilon}, \qquad k \in \mathbb{R},\end{aligned} \tag{59}$$

$$\mu_-(-\mathrm{i}\kappa_l) = 1 - \sum_{j=1}^{N} C_j \mathrm{e}^{-2\kappa_j x} \frac{\mu_-(-\mathrm{i}\kappa_j)}{\mathrm{i}\kappa_l + \mathrm{i}\kappa_j} + \frac{1}{2\pi\mathrm{i}} \sum_{\pm} \int_{C_\pm} F_\pm(z)\mu_\pm(-\bar{z}) \frac{\mathrm{e}^{\mathrm{i}[qy+2z_R x]}}{z + \mathrm{i}\kappa_l} \,\mathrm{d}z \wedge \mathrm{d}\bar{z}$$
$$+ \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}z \frac{\rho(z)\mathrm{e}^{2\mathrm{i}zx}\mu_-(-z)}{z + \mathrm{i}\kappa_l}, \qquad (60)$$

which, along with (52), defines a closed system of equations to recover the function $\mu_-(x, y, k)$.

2. After restoring the initial coordinates (i.e. returning from the primed to unprimed coordinates) u is given by

$$u(x, y) = \frac{\partial}{\partial x}\Bigg[2\mathrm{i} \sum_{1}^{N} C_j \mathrm{e}^{-2\kappa_j(x-vy)} \mu_-(x - vy, y, -\mathrm{i}\kappa_j)$$
$$- \frac{1}{\pi} \sum_{\pm} \int_{C_\pm} F_\pm(z)\mu_\pm(x - vy, y, -\bar{z})\mathrm{e}^{\mathrm{i}[qy+2z_R(x-vy)]} \,\mathrm{d}z \wedge \mathrm{d}\bar{z}$$
$$- \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}z\rho(z)\mathrm{e}^{2\mathrm{i}z(x-vy)} \mu_-(x - vy, y, -z)\Bigg]. \qquad (61)$$

□

Whenever a discrete spectrum is present, i.e. when the set $\{k_j \equiv \mathrm{i}\kappa_j, \kappa_j \in R^+\}_{j=1,\ldots,N}$ of (simple) zeroes of $a(k)$ is not empty, the function ν_+ has simple poles at $k = \mathrm{i}\kappa_j$. This suggests that the IP (52) might have a double pole, *viz* a nonintegrable singularity, rendering the integral of the IP divergent. We next give a detailed analysis of the behaviour of F_+ at the singularities and show that they are smoother.

Proposition 16. *The poles in the IP are not double, namely one has that*

$$\lim_{k\to\mathrm{i}\kappa_j} F_1(k_R, k_I) = 0, \qquad (62)$$

where

$$F_1(k) \equiv \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{-\mathrm{i}qy} U\nu_1(x, y, k)\psi_+(x, -\bar{k})$$

and $\nu_1 \equiv a(k)\nu_+(x, y, k) \equiv a(k)\mu_+(x, y, k)\mathrm{e}^{-\mathrm{i}kx}$.

Proof. In what follows we can suppose generically that $k_R \neq 0$ since (i) the symmetric definition of the $\bar{\partial}$ derivative yields $\lambda = 0$ on $\{k_R = 0\}$, and (ii) $\{k_R = 0\}$ is a set of zero measure. Using that (see (45)) $\nu_1 = \phi_+(x, k) + e(x, y, k)$ where the properties of $e(x, y, k)$ are given in (46) and (48), and that $\phi_+(x, k)$ is a solution of the SO $A(x, \partial_x, k) \equiv \partial_{xx} + k^2 + u_\infty(x)$ we have

$$F_1(k) = \int \mathrm{d}x\,\psi_+(x, -\bar{k}) \int \mathrm{d}y\,\mathrm{e}^{-\mathrm{i}qy}[\partial_y - A]e(x, y, k).$$

Note that the integral involves well-defined terms; indeed, one has

$$\psi_+(x, -\bar{k}) = \mathrm{e}^{-\mathrm{i}\bar{k}x}\tilde{\psi}_+(x, -\bar{k}),$$

$$e(x, y, k) = \mathrm{e}^{-\mathrm{i}kx}\tilde{e}_1 + \mathrm{e}^{\mathrm{i}\bar{k}x}\tilde{e}_2,$$

where if $k_R \neq 0$ and $r^2 \equiv x^2 + y^2$, then as $r \to \infty$

$$\tilde{e}_1(x, y, k) = -\frac{A(k)\tilde{\phi}_+(x, k)}{2ky + \mathrm{i}x} + O\left(\frac{1}{r^2}\right);$$

$$\tilde{e}_2(x, y, k) = -\tilde{\phi}_+(x, -\bar{k})\frac{F(k)a(k)}{a(-\bar{k})}\frac{\mathrm{e}^{\mathrm{i}qy}}{2\bar{k}y - \mathrm{i}x} + O\left(\frac{1}{r^2}\right),$$

$\tilde{\psi}_+$, $\tilde{e}_1$, e_2 are bounded; hence so is $\psi_+(x, -\bar{k})\mathrm{e}^{-\mathrm{i}qy}e(x, y, k)$.

We obtain via integration by parts and use of the decay of $e(x, y, k)$ as $|y| \to \infty$ that

$$\int \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}qy}\partial_y e(x, y, k) = (\mathrm{e}^{-\mathrm{i}qy}e(x, y, k))_{y=-\infty}^{y=\infty} + \mathrm{iq}\int \mathrm{dy}\, \mathrm{e}^{-\mathrm{iqy}}\mathrm{e(x, y, k)}$$
$$= \mathrm{i}q\int \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}qy}e(x, y, k),$$

which goes to zero as k approaches $\mathrm{i}\kappa_j$.

Likewise, we have by integration by parts that

$$\int \mathrm{d}x\, \psi_+(x, -\bar{k})A\int \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}qy}e(x, y, k) = (\psi(x, -\bar{k})\partial_x r(x, k) - \partial_x\psi(x, -\bar{k})r(x, k))_{x=-\infty}^{x=\infty}$$
$$+ \int r(x, k)[k^2 - \bar{k}^2 + A(x, \partial_x, -\bar{k})]\psi(x, -\bar{k})\,\mathrm{d}x,$$

where

$$r(x, k) \equiv \int \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}qy}e(x, y, k).$$

Note that $A(x, \partial_x, -\bar{k})\psi(x, -\bar{k}) = 0$.

Hence, letting k approach $\mathrm{i}\kappa_j$ along any direction with $k_R \neq 0$ and since $\psi(x, \mathrm{i}\kappa_j)$, $\partial_x\psi(x, \mathrm{i}\kappa_j)$ are exponentially decaying as $x^2 + y^2 \to \infty$ we have

$$\lim_{k\to \mathrm{i}\kappa_j, k_R\to 0}(\psi(x, -\bar{k})\partial_x r(x, k) - \partial_x\psi(x, -\bar{k})r(x, k))_{x=-\infty}^{x=\infty} = 0,$$

$$\lim_{k\to \mathrm{i}\kappa_j, k_R\to 0} F_1(k) = 0. \qquad \square$$

7. Temporal evolution and line solitons

KPII is obtained as the compatibility of (2) and

$$M\Psi \equiv \left(\partial_t + 4\partial_{xxx} + 6u\partial_x + 3u_x + 3(\partial_y\partial_x^{-1}u) + 4\left(\frac{v}{2} + \mathrm{i}k\right)^3\right)\Psi = 0. \tag{63}$$

The temporal evolution of the scattering data follows in the usual way by taking the $\bar{\partial}$ derivative and substituting the relevant eigenfunctions in the time operator evolution M. The procedure is standard, hence we need not dwell on the derivation. The temporal evolution of the scattering data is given by

$$F_\pm(k, t) = F_\pm(k, 0)\mathrm{e}^{4[(v/2-\mathrm{i}\bar{k})^3-(v/2+\mathrm{i}k)^3]t},$$

$$\rho(k, t) = \rho(k, 0)\mathrm{e}^{4[(v/2-\mathrm{i}k)^3-(v/2+\mathrm{i}k)^3]t}; \qquad C_j(t) = C_j(0)\mathrm{e}^{4[(v/2+\kappa_j)^3-(v/2-\kappa_j)^3]t}. \tag{64}$$

Hence, the solution $u(x, y, t)$ of the KPII equation is obtained from equations (52) and (53) after the time dependence' of $F_\pm$, ρ, C_j are inserted. It follows from these equations that

$$\mu(x, y, t, k) = 1 + \sum_{j=1}^{N} C_j \mathrm{e}^{-2\kappa_j x + 4[(v/2+\kappa_j)^3 - (v/2-\kappa_j)^3]t} \frac{\mu_-(-\mathrm{i}\kappa_j)}{k - \mathrm{i}\kappa_j}$$
$$+ \frac{1}{2\pi\mathrm{i}} \sum_{\pm} \int_{C_\pm} F_\pm(z)\mu_\pm(-\bar{z}) \frac{\mathrm{e}^{\mathrm{i}[qy+2z_R x+4[(v/2-\mathrm{i}\bar{k})^3-(v/2+\mathrm{i}k)^3]t]}}{z-k}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}$$
$$+ \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}z_R \frac{\rho(z_R)\mathrm{e}^{2\mathrm{i}zx+4[(v/2-\mathrm{i}k)^3-(v/2+\mathrm{i}k)^3]t}\mu_-(-z_R)}{z_R - k}, \tag{65}$$

$$u = \frac{\partial}{\partial x}\Bigg[2\mathrm{i} \sum_{1}^{N} C_j \mathrm{e}^{-2\kappa_j x+4[(v/2+\kappa_j)^3-(v/2-\kappa_j)^3]t} \mu_-(-\mathrm{i}\kappa_j)$$
$$- \frac{1}{\pi} \sum_{\pm} \int_{C_\pm} F_\pm(z)\mu_\pm(-\bar{z}) \mathrm{e}^{\mathrm{i}[qy+2z_R x+4[(v/2-\mathrm{i}\bar{k})^3-(v/2+\mathrm{i}k)^3]t]}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}$$
$$- \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}z_R \rho(z_R) \mathrm{e}^{2\mathrm{i}z_R x+4[(v/2-\mathrm{i}k)^3-(v/2+\mathrm{i}k)^3]t} \mu_-(-z_R)\Bigg]. \tag{66}$$

When the continuous scattering data are all zero—$F_\pm(k) = \rho(k) = 0$—the equations to recover the eigenfunction μ in primed coordinates (reminding the reader that $x' = x - vy$) are

$$\mu(-\mathrm{i}\kappa_l) = 1 - \sum_{1}^{N} C_j \mathrm{e}^{-2\kappa_j x'} \frac{\mu_-(-\mathrm{i}\kappa_j)}{\mathrm{i}(\kappa_j + \kappa_l)}.$$

Solving this system in the same way as for KdV one finds that the solution is given in terms of these data by

$$u(x', t) = 2\frac{\mathrm{d}^2}{\mathrm{d}x'^2} \log\det F(x', t),$$

where the $N \times N$ matrix $(F_{lj})_{N\times N}$ is defined as

$$F_{lj} = \delta_{lj} + C_l \frac{\mathrm{e}^{-(\kappa_j+\kappa_l)x'}}{\mathrm{i}(\kappa_l + \kappa_j)}.$$

If we restore the original coordinates and introduce the temporal evolution we obtain

$$u(x, y, t) = 2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \log\,\det F(x - vy, t),$$

where the $N \times N$ matrix $(F_{lj})_{N\times N}$ is defined by

$$F_{lj} = \delta_{lj} - \mathrm{i}C_l \frac{\mathrm{e}^{-(\kappa_j+\kappa_l)(x-vy)+8\kappa_j(\kappa_j^2+3v^2 4)t}}{\kappa_j + \kappa_l}.$$

The solutions are line solitons, all of them moving along the same direction. In particular, with $N = 1$, $\kappa_1 \equiv \kappa$, $x_0 = \frac{1}{2}\kappa \log(C_1(0)/2\mathrm{i}\kappa)$

$$u(x, y, t) = 2\kappa^2 \mathrm{sech}^2\kappa[(x - vy) - (4\kappa^2 + 3v^2)t - x_0].$$

Returning to the general case, we note that the following important consequences are clear from the general solution of the equation (65).

Proposition 17.

(i) *The number of solitons that develop from KPII evolution with initial data* $u(x, y, 0) = u_\infty(x - vy) + U(x, y)$ *is the same as the number of solitons that develop from KdV evolution with initial data* $u(x, 0) = u_\infty(x)$.

(ii) *The line solitons are nonlinearly stable against decaying perturbations in the plane* $U(x, y)$ *satisfying (43). More generally, KPII solutions of the form* $u(x, y, t) = \varphi(x - vy, t)$ *are nonlinearly stable against decaying perturbations in the plane.*

Remarks.

1. Note that (ii) confirms the results of [7] where the stability of these solitons against a particular kind of perturbation was suggested via a perturbative approach (see also [17], pp 259–260 and the seminal paper [1]).
2. The stability of line solitons against random perturbations has been recently studied (cf [44]).

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