Dynamics of Lump Solutions in a 2 + 1 NLS Equation

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We derive a class of localized solutions of a 2+1 nonlinear Schrödinger (NLS) equation and study their dynamical properties. The ensuing dynamics of these configurations is a superposition of a uniform, "center of mass" motion and a slower, individual motion; as a result, nontrivial scattering between humps may occur. Spectrally, these solutions correspond to the discrete spectrum of a certain associated operator, comprised of higher-order meromorphic eigenfunctions.

1. Introduction

In this paper we derive and study the dynamical properties of a class of solutions of a 2+1-dimensional differential equation with boundary conditions, namely,

$$iu_t + u_{xx} + 2u\partial_x \int_{-\infty}^{y} \left(1 - |u|^2\right) dy' = 0, \lim_{r\to\infty} |u|(x, y, t) = 1 \tag{1}$$

where $u(x, y, t)$ is a complex function, depending on three real variables x, y, t and $r^2 \equiv x^2 + y^2$. Note that the reduction to the manifold $x = y$ yields the nonlinear Schrödinger (NLS) equation and hence Equation (1) generalizes the latter to the plane. Recall that the NLS equation was formulated in 1968 by Zakharov [1] as an equation describing dynamics of waves in deep water with surface tension; a year later Benney and Roskes [2] showed that the derivation

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carried over to finite-depth water. NLS has also been shown to describe pulse transmission in optical fibers [3], which further underlines the interest of this equation. The natural issue of generalizing it to 2+1-dimensions was addressed in the context of shallow water waves by Benney and Roskes [2] and Davey and Stewartson [4] who wrote down the DS equation, namely $iu_t + u_{yy} + \sigma u_{xx} + u(|u|^2 - m) = 0$, $\sigma m_{yy} - m_{xx} = -2\partial_{xx}|u|^2$, $\sigma = \pm 1$.

Equation (1) appears naturally as another interesting *integrable* generalization of NLS while its ulterior simplicity and potential physical interest makes it a natural candidate for study. We note that the related equation corresponding to decaying boundary conditions, viz.

$$iu_t + u_{xx} - 2u\partial_x \int_{-\infty}^{y} dy'|u|^2 = 0 \tag{2}$$

was first brought to attention and proven to be integrable by Fokas [5].

Actually, both (1) and (2) are particular cases of the more general system

$$m_y + |u|^2 = 0, \;\; iu_t + u_{xx} + 2um_x = 0 \tag{3}$$

corresponding to definite boundary conditions and a given prescription for ∂_y^{-1}.

Equations (3) can be written as the compatibility of a linear pair of operators. The spectral theory of the associated spatial operator under *certain* boundary conditions, has already been described in [6] in connection with the solution of the Cauchy problem of a certain different nonlinear equation. From this analysis it follows that for potentials satisfying $|u|^2 - 1 \in L_1(\mathbb{R}^2)$ no discrete spectrum exists and only continuous spectrum does. Nevertheless, as we show here, the operator has also a discrete spectrum, corresponding to potentials that can be written as $u = 1 + \tilde{u}$ where $\tilde{u}$ is a regular, *weakly decaying* function. These configurations shall be termed the lumps. While both Equations (1) and (2) are linearizable and the initial value problem solvable by means of inverse scattering transform method (IST), it is expected that only the former possesses lump solutions.

With more generality, lump solutions are localized wave configurations that decay rationally to an asymptotic value and move with uniform velocity. The basic asymptotic dynamics consists of a uniform motion; in addition, lumps display no scattering upon interaction, just a parallel shift on the asymptotic motion. We expect for lumps to play in 2+1 dimensions a similar role to solitons in 1+1-dimensions and hence that general initial data will evolve for sufficiently long time into a sum of lumps. In the light of such a fundamental role as basic building blocks they have been extensively studied in the last years. They were first found for the Kadomtsev–Petviashvili I (KPI) equation. The spectral interpretation in terms of a discrete spectrum was clarified in [7] (see also [8]. For a description of the KP equation and its physical origins see [9]). Subsequently, lumps have been found in other integrable equations like Davey–Stewartson II (DSII) (see [10]) and the 2+1-Toda lattice, see [11].

Surprisingly, the above is far from being an accurate description of the generic behavior of localized pulses of integrable equations: it turns out that KPI posses a much richer discrete spectrum corresponding to a class of localized, real and regular solutions with weak decay at infinity that have, however, a nontrivial asymptotic dynamics. Even though the simplest of these solutions has been known for a long time [12] the associated nontrivial scattering was overlooked by the soliton community up until 1995 [13, 14]. Clarification via spectral analysis and the derivation of the general class of those solutions was undertaken in [15]. These new, nonstandard lumps were found to be associated to a new discrete spectrum of the time dependent Schrödinger operator corresponding to meromorphic eigenfunctions with poles of higher multiplicity and/or to what we term nonstandard pole divisors, defined by a relationship between Laurent coefficients (LC) that generalizes that of [7] and which, in the spirit of the ideas of [15] and [16], are associated to integer winding numbers. The extension of these ideas and solutions to DSII equation via spectral analysis of the Dirac operator on the plane has been considered in [16]. (We also note that both KPI and KPII possess, in addition, other localized, *nondecaying* solutions like line solitons. The solution of the Cauchy problem for KPII in such a background is considered in [17, 18, 19]). An updated and comprehensive account of all these ideas is given in [20]).

In this paper we perform a related analysis for Equation (1) and show that there exits a discrete spectrum that encompasses a whole manifold of smooth, rationally "decaying" lump configurations. We find that they are associated with higher-order pole meromorphic eigenfunctions of a similar discrete spectrum. Particular attention is paid to describing the dynamics and scattering properties of some of these configurations. Lump solutions of the kind considered here have also been obtained via these direct methods in [21]. Note also that in [22, 23] and [24] it was proven that Equations (1) satisfy Painleve's test; in addition some special solutions, like line solitons, lumps and dromion solutions, were found. The spectral theory of these configurations is to our knowledge an open problem. The relation with the generalized dispersive wave equation via Miura transformations was considered in [25].

The organization of the paper is as follows. The associated pair of linearizing operators is given in Section 2. Section 3 is devoted to obtaining certain relations that meromorphic eigenfunctions must satisfy to be eigenfunctions of the Lax pair. In Section 4 we study both lump solutions with standard and nonstandard dynamics and their properties. In the latter case we find that the dynamics is a superposition of a uniform "center of mass" motion, and a individual, lump depending motion, that behaves as $|t|^q$ with $q < 1$ (Actually, $q = \frac{1}{2}$ for the case considered here.). From a dynamical perspective, the latter is the only nontrivial motion. Frontal collision of lumps can be expected as a result of which lumps *may* scatter off in a nontrivial way. We consider here the simplest nontrivial cases. A more detailed report of these general lump solutions will be given elsewhere.

2. Linear problem

As we have already pointed out Equation (1) arises as the compatibility of a pair of operators, see [5, 21, 24]. Under the boundary conditions[1] (BC) $\lim_{r\to\infty}|u|(x, y, t) = 1$ a convenient form of the Lax pair, depending on a complex spectral parameter k, is given by the following pair of linear operators L, M:

$$L\mu \equiv \mu_{xy} + \frac{1}{k}\mu_x + \left(k - \frac{u_x}{u}\right)\mu_y + \left(1 - \frac{1}{k}\frac{u_x}{u} - |u|^2\right)\mu = 0 \qquad (4)$$

$$M\mu \equiv \mu_{xx} + i\mu_t + 2k\mu_x + 2n\mu = 0. \qquad (5)$$

This Lax pair, which already incorporates a spectral parameter $k \in \mathbb{C}$, is more convenient than those used in [5] and [24]. We use $n(x, y, t) \equiv \partial_x \int_{-\infty}^{y} (1 - |u|^2)\, dy'$. The existence of an eigen function $\mu(x, y, t, k)$ normalized to 1 as $|k| \to \infty$ is assumed in the sequel. Then, Equations (4) and (5) implies that $\mu(k)$ has an asymptotic expansion in the vicinity of infinity with coefficients $\mu^{(j)}(x, y, t)$, viz.

$$\mu = \mu^{(0)} + \frac{\mu^{(1)}}{k} + \frac{\mu^{(2)}}{k^2} + \cdots, \quad |k| \to \infty \qquad (6)$$

$$\text{where } \mu^{(0)} \equiv 1, \ \ |u|^2 = 1 + \partial_y \mu^{(1)}. \qquad (7)$$

This expression permits to recover the "physical amplitude" $|u|^2$. The phase of the potential u involves considering higher-order terms and will not be given here.

3. Discrete spectrum and meromorphic eigenfunctions

3.1. General expansions

Here we show the existence of meromorphic eigenfunctions with poles of higher-order multiplicity. These singular wave functions correspond to potentials $u(x, y, t)$ such that $u - 1$ is a regular function with slow decay at infinity. The class of all such functions will define the discrete spectrum.

[1]The situation corresponding to general BCs and the bearing of this on the conserved quantities will not be considered here. We also remark that the BC $\lim_{r\to\infty}|u|(x, y, t) = 1$ implies the need of constraints. See [26] for the implications of this fact for KP equation

We shall suppose that a meromorphic solution of both Equations (4) and (5) exists with a finite or denumerable number of poles $\{k_n\}$. Around any pole $k = k_1$, $\mu(k)$ has a local Laurent expansion

$$\mu(k) = \mu^{k_1)}_{\text{sing.}}(k) + \mu^{k_1)}_{\text{reg.}}(k), \tag{8}$$

$$\text{where} \qquad \mu^{k_1)}_{\text{reg.}}(k) \equiv \sum_{r=0}^{\infty} \nu_r (k-k_1)^r, \quad \mu^{k_1)}_{\text{sing.}}(k) \equiv \sum_{r=1}^{m} \frac{\phi_r}{(k-k_1)^r}, \tag{9}$$

and $\phi_r = \phi_{k_1,r}(x, y, t)$ and $\nu_r = \nu_{k_1,r}(x, y, t)$ are the LC in the local Laurent expansion around k_1. Here and elsewhere a subscript is used, whenever appropriate, to stress the dependence of the former on the pole. Notice that $\mu^{k_1)}_{\text{sing.}}(k)$ is also termed the principal part or the pole divisor. In the next section we study how they relate.

Unlike what happens in the regular case, where the eigenfunction is fixed by the corresponding $\bar{\partial}$-problem, when singularities exist the inverse problem does not fix uniquely the singular part. Additional information—relating different coefficients of the poles divisor—is required. As we see, the different LC must be related in certain, nonunique, ways.

In this section, we consider examples of meromorphic eigenfunctions related to nonsingular potentials and determine different relationships between coefficients of the poles divisor.

3.2. Simple poles

We shall first suppose that $\mu \equiv \mu(x, y, t, .)$ is a meromorphic eigenfunction with a simple pole at k_1. In this case Equation (8) reads

$$\mu(x, y, t, k) = \frac{\phi(x, y, t)}{k - k_1} + \mu^{k_1)}_{\text{reg.}}(x, y, t, k). \tag{10}$$

Insertion of the above into Equations (4) and (5) yields as $k \to k_1$ that ϕ and $\nu \equiv \nu_0$ must satisfy

$$\phi_{xy} + \frac{1}{k_1}\phi_x + \phi_y\left(k_1 - \frac{u_x}{u}\right) + \left(1 - \frac{1}{k_1}\frac{u_x}{u} - |u|^2\right)\phi = 0 \tag{11}$$

$$\begin{aligned} &\nu_{xy} + \frac{1}{k_1}\nu_x + \nu_y\left(k_1 - \frac{u_x}{u}\right) + \nu\left(1 - \frac{1}{k_1}\frac{u_x}{u} - |u|^2\right) \\ &\quad + \left(\phi_y - \frac{1}{k_1^2}\phi_x + \frac{1}{k_1^2}\frac{u_x}{u}\phi\right) = 0 \end{aligned} \tag{12}$$

$$\text{and } \phi_{xx} + i\phi_t + 2k_1\phi_x + 2n\phi = 0, \quad \nu_{xx} + i\nu_t + 2k_1\nu_x + 2n\nu + 2\phi_x = 0. \tag{13}$$

It turns out that the latter equations do not relate uniquely the coefficients. It is possible to obtain infinitely many relationships compatible with the above structure. Here we consider a few of them.

We first suppose that ν and ϕ are related as $\nu = f(x, y, t)\phi$. Then, Equations (11)–(13) read

$$\left(f_{xy} + f_x + k_1 f_y - \left(f_y - \frac{1}{k_1^2}\right)\frac{u_x}{u}\right)\phi + \phi_y(f_x + 1) + \phi_x\left(f_y - \frac{1}{k_1^2}\right) = 0 \tag{14}$$

$$\phi(f_{xx} + if_t + 2k_1 f_x) + 2\phi_x(f_x + 1) = 0. \tag{15}$$

We can satisfy these equations by requiring f to solve the following system of *constant* differential equations

$$f_{xy} + f_x + k_1 f_y = f_y - \frac{1}{k_1^2} = 0,\ f_{xx} + if_t + 2k_1 f_x = f_x + 1 = 0. \tag{16}$$

It follows that

$$f = -x - 2ik_1 t + \frac{y}{k_1^2} + \gamma_1, \quad \text{where } \gamma_1 \equiv \gamma_R + i\gamma_I \in \mathbb{C}. \tag{17}$$

Therefore, we have obtained that if ϕ solves both Equations (11) and (the first of) (13) and also if the relationship between LC

$$\nu_0 = \left(-x - 2ik_1 t + \frac{y}{k_1^2} + \gamma_1\right)\phi \tag{18}$$

is satisfied, then (10) satisfies locally both (4) and (5). Whenever Equation (18) holds we say that we have an eigenfunction with simple poles and standard divisors. Such a relation was first established for KPI in [7].

We next study the possibility of having nonstandard pole divisors. To this end we posit the existence of meromorphic eigenfunctions $\mu \equiv \mu(x, y, t, .)$ with simple poles, where a linear relationship between the three first LCs ϕ, ν_0 and *also* ν_1 obtains. We first note that, by going to next order, one obtains the hierarchy of Equations (11)–(13) along with

$$(\partial_{xx} + i\partial_t + 2k\partial_x + 2n)\nu_1 + 2\partial_x\nu_0 = 0 \tag{19}$$

$$\begin{aligned}\nu_{1xy} + \frac{1}{k_1}\nu_{1x} + \left(k_1 - \frac{u_x}{u}\right)\nu_{1y} + \left(1 - \frac{1}{k_1}\frac{u_x}{u} - |u|^2\right)\nu_1 - \frac{\nu_{0x}}{k_1^2}\\ + \nu_{0y} + \frac{u_x}{k_1^2 u}\nu_0 = 0.\end{aligned} \tag{20}$$

We shall assume next that Equation (18) no longer holds; instead, a linear relationship between LCs of the form $\nu_1 = f\nu_0 + g\phi$ applies. Here f, g are certain functions to be determined. We find that Equation (19) is satisfied if

$$f_{xx}\nu_0 + g_{xx}\phi + 2f_x\nu_{0x} + 2g_x\phi_x + i(f_t\nu_0 + g_t\phi) + 2k_1(f_x\nu_0 + g_x\phi) + 2\nu_{0x} - 2f\phi_x = 0. \tag{21}$$

This equation will be satisfied if f and g solve the linear system of PDE's

$$f_{xx} + if_t + 2k_1f_x = 0,\ f_x + 1 = 0, \quad g_{xx} + ig_t + 2k_1g_x = 0, g_x = f. \tag{22}$$

It follows first that f must be given, again, by (17). To solve the equations. for g note that in the new coordinates $y' = y$, $t' = t$, $x' = f(x, y, t)$ they read $g_f = -f$, $g_{ff} + ig_t = 0$ which implies that $g = -\frac{f^2}{2} - it + L(y)$. The unknown function $L(y)$ is determined substituting $\nu_1 = f\nu_0 + g\phi$ into Equation (20) to find

$$\nu_1 = f\nu_0 + g\phi \equiv \left(-x - 2ik_1t + \frac{y}{k_1^2} + \gamma\right)\nu_0 - \left(\frac{f^2}{2} + it + \frac{y}{k_1^3} - \delta\right)\phi \tag{23}$$

where γ and δ are complex constants. Thus, Equation (23) gives a different relationship between the three first LCs ϕ, ν_0, and ν_1 of simple pole eigenfunctions which is also compatible with the Lax pair. In this case we say that we have simple poles of index 2.

3.3. Double poles

In this section, we assume the existence of a singular eigenfunction with a double pole at some point $k = k_1$. As we shall see admissible wave functions require that the coefficients of the principal and regular parts are related in a certain way. Suppose then that around a pole $k = k_1$, $\mu(k)$ has a local Laurent expansion with pole divisor

$$\mu^{k_1)}_{\text{sing.}}(k) = \frac{\phi}{k - k_1} + \frac{\psi}{(k - k_1)^2} \tag{24}$$

where we use the notation $\phi_{k_1,2} \equiv \psi$, $\phi_{k_1,1} \equiv \phi$ and also $\nu_{k_1,0} \equiv \nu$. By letting $k \to k_1$ in Equations (4) and (5) we find that the main coefficients must satisfy $\psi_{xx} + i\psi_t + 2k_1\psi_x + 2n\psi = 0$, and also

$$\psi_{xy} + \frac{1}{k_1}\psi_x + \left(k_1 - \frac{u_x}{u}\right)\psi_y + \left(1 - \frac{1}{k_1}\frac{u_x}{u} - |u|^2\right)\psi = 0 \tag{25}$$

$$\phi_{xy} + \frac{1}{k_1}\phi_x + \left(k_1 - \frac{u_x}{u}\right)\phi_y + \left(1 - \frac{1}{k_1}\frac{u_x}{u} - |u|^2\right)\phi - \frac{1}{k_1^2}\psi_x + \psi_y + \frac{1}{k_1^2}\frac{u_x}{u}\psi = 0 \tag{26}$$

$$\nu_{xy}+\frac{1}{k_1}\nu_x+\left(k_1-\frac{u_x}{u}\right)\nu_y+\left(1-\frac{1}{k_1}\frac{u_x}{u}-|u|^2\right)\nu-\frac{1}{k_1^2}\phi_x+\phi_y$$
$$+\frac{1}{k_1^2}\frac{u_x}{u}\phi+\frac{1}{k_1^3}\psi_x-\frac{1}{k_1^3}\frac{u_x}{u}\psi=0 \tag{27}$$

$$\phi_{xx}+i\phi_t+2k_1\phi_x+2n\phi+2\psi_x=0 \tag{28}$$

$$\nu_{xx}+i\nu_t+2k_1\nu_x+2n\nu+2\phi_x=0. \tag{29}$$

We look to satisfy these equations by assuming that the coefficients satisfy the following linear relationships

$$\phi=f\psi \text{ and } \nu=h\psi \tag{30}$$

where f, h depend on x, y, t. As above, from (26, 28) we first find f to be given by Equation (17). Next, from (27, 29) h must satisfy

$$h_{xx}+ih_t+2k_1h_x+2f_x=0,\quad h_x+f=0 \tag{31}$$

or, alternatively, $h_f-f=0, ih_t=1+2k_1f$. It follows that $h=\frac{f^2}{2}-it+\beta(y)$, for some function $\beta(y)$. Inserting this into Equation (27) several terms cancel and we see that it will be satisfied provided we require the simple condition $\beta'+\frac{1}{k_1^3}=0$. Thus, the assumption (30) is compatible with the given analytic structure provided f and h are given by

$$f=-x-2ik_1t+\frac{y}{k_1^2}+\gamma \text{ and } h=\frac{f^2}{2}-it-\frac{y}{k_1^3}+\delta \tag{32}$$

where γ, δ are complex constants. Whenever such a possibility occurs we say that the eigenfunction has *double poles with index two*.

However, it turns out that the relationship (30) is not the only one possible for eigenfunctions with double poles: again, the assumed analytic structure does not fix uniquely the way LCs are related. A different admissible situation corresponds to having coefficients related as

$$\phi=f\psi \text{ and } \nu_1=A(x,y,t)\nu_0+B(x,y,t)\psi. \tag{33}$$

It can be proven that if $f=-x-2ik_1t+\frac{y}{k_1^2}+\gamma$, $A=f+\varrho$ and

$$B=\frac{-1}{3}\left(f^3+3/2\varrho f^2\right)+i\varrho t+\left(\frac{\varrho}{k_1^3}+\frac{1}{k_1^4}\right)y+\varpi \tag{34}$$

where γ,ϱ,ϖ are complex constants—then the analytic structure (24) with (33) corresponds to an eigenfunction of operators (4, 5). We skip the details here.

4. Algebraically decaying solutions

Here we use the results of Section 3 to determine several classes of algebraically decaying solutions of Equation (1); they correspond to concrete meromorphic eigenfunctions whose LCs satisfy adequate relationships.

4.1. Properties of the standard lump solution

We first assume that $\mu(k)$ is a meromorphic eigenfunction that has the simple representation

$$\mu(k) = 1 + \left(\frac{\phi_1}{k - k_1} + \frac{\phi_{\bar{1}}}{k + \bar{k}_1}\right) \tag{35}$$

i.e., we suppose that $\mu(k)$ has simple poles at $k_1 \equiv a + ib, -\bar{k}_1$ and corresponding residues $\phi_1, \phi_{\bar{1}}$. We set $k_1 \equiv a + ib$ and suppose $a \neq 0$; we also require that every pole divisor is standard and hence satisfies Equation (18) with constants k_1, γ_1, and $-\bar{k}_1, \gamma_{\bar{1}}$ and that $\gamma_{\bar{1}} = \bar{\gamma}_1 \equiv \gamma_R - i\gamma_I \in \mathbb{C}$. Then, Equation (18) yields

$$1 + \frac{\phi_{\bar{1}}}{k_1 + \bar{k}_1} = f_1\phi_1, \;\; 1 - \frac{\phi_1}{k_1 + \bar{k}_1} = \bar{f}_1\phi_{\bar{1}}. \tag{36}$$

Solving this system of linear equations and using Equation (7) we find the potential:

$$|u|^2 = 1 + \partial_y(\phi_1 + \phi_{\bar{1}}) = 1 - \partial_{yx}\log\tau \tag{37}$$

where the τ-function is just the determinant of the matrix associated to this system:

$$\tau \equiv f_1\bar{f}_1 + \frac{1}{4a^2}. \tag{38}$$

Obviously, this solution is smooth and has a rational decrease to the background value $u_\infty = 1$.

Remark. We expect that in a general situation nonsingular solutions are associated to a discrete spectrum with pairs of poles $k_1, k_{\bar{1}} = -\bar{k}_1$ and appropriate norming constants satisfying, say, $\gamma_{\bar{1}} = \bar{\gamma}_1$. Understanding why this must hold poses an interesting question.

To study further dynamical properties of the solution it is useful to consider new coordinates (X, Y) defined by

$$X = x - Vt - \tilde{\gamma}, Y = y + bV^2t + \frac{\gamma_I}{\rho} \tag{39}$$

where we rename parameters as

$$V = \frac{a^2+b^2}{b}, \beta = \frac{a^2-b^2}{(a^2+b^2)^2}, \ \rho = -\frac{2ab}{(a^2+b^2)^2}, \tilde{\gamma} \equiv \gamma_R - \frac{a^2-b^2}{2ab}\gamma_I. \tag{40}$$

In this frame, we have $\tau = (X-\beta Y)^2 + \rho^2 Y^2 + \frac{1}{4a^2}$ and the potential reads

$$|u|^2 = 1 - 2\beta\left((X-\beta Y)^2 + \frac{\rho^2 Y}{\beta}(\beta Y - 2X) - \frac{1}{4a^2}\right)\frac{1}{\tau^2}. \tag{41}$$

Thus, solution (41) is a traveling wave which will be called the standard 1-lump solution of Equation (1). Further properties of the former solution are of interest. It displays a maxima structure far richer than what is common for both lump and soliton solutions. Inspection shows that critical points are located at $(X, Y) = (0, 0)$ and

$$X = \pm\frac{1}{4ab}\sqrt{3b^2-a^2}, \quad Y = \pm\frac{a^2+b^2}{4ab}\sqrt{3b^2-a^2} \tag{42}$$

$$Y = \pm\frac{a^2+b^2}{4a^2}\sqrt{3a^2-b^2}, \ X = \mp\frac{1}{4a^2}\sqrt{3a^2-b^2}. \tag{43}$$

These points are all candidates to maxima and minima; the number of them will vary depending on the values of the parameters. The parameter space is a two-dimensional plane deprived of the straight line $a = 0$. To describe the situation in the general case we restrict, with no loss of generality, to the first quadrant on the parameter space. The situation varies according to which of the regions

$$C_1 \equiv \left\{(a, b) : a \le \frac{b}{\sqrt{3}}\right\}, C_2 \equiv \{(a, b) : b/\sqrt{3} \le a \le \sqrt{3}b\},$$
$$C_3 \equiv \{(a, b) : 0 \le \sqrt{3}b \le a\}$$

do parameters belong. If (a, b) is in the interior of C_2 then two of the four points described by (42, 43) are maxima and the other two are minima. Point $(X, Y) = (0, 0)$ is a saddle point between the former at which $|u| = |3a^2 - b^2|/(a^2 + b^2)$. In the particular case when $a = b$ the potential's amplitude reads simply

$$\upsilon \equiv |u|^2 - 1 = \frac{XY}{a^4\left(X^2 + \frac{Y^2}{4a^4} + \frac{1}{4a^2}\right)^2}. \tag{44}$$

It has two symmetric maxima located at $p = \pm\sqrt{2}(\frac{1}{4a}, \frac{a}{2})$ at which $\upsilon \equiv |u|^2 - 1 = 1$ while minima are to be found at the mirror images points at which $\upsilon = -1$.

The boundary point $a = \sqrt{3}b$ corresponds to a configuration having a maximum $(X, Y) = (0, 0)$ at which $\upsilon = 3$; the minimum lump's amplitude

$\upsilon = -1$ is to be found at the points $(X, Y) = \pm\frac{\sqrt{8}}{12b}(1, -4b^2)$. At the other boundary point $b = \sqrt{3}a > 0$ the situation is opposite and is rather reminiscent of dark solitons; it corresponds to a configuration having minimum amplitude $\upsilon = -1$ at the origin and maximum amplitude $\upsilon = 3/25$ at the points $(X, Y) = \pm\frac{\sqrt{8}}{4\sqrt{3}a}(1, 4a^2)$.

Thus, generically the configuration is localized on an entire region containing several maxima.

The choice $b = 0$ yields a degenerate solution that deserves some attention. In this case letting $\hat{t} \equiv t - \gamma_I/2a, z \equiv x - \frac{y}{a^2} + \gamma_R$ the solution and the maximum amplitude read

$$\upsilon \equiv |u|^2 - 1 = \frac{2}{a^2}\frac{-z^2 + 4a^2\hat{t}^2 + \dfrac{1}{4a^2}}{\left(z^2 + 4a^2\hat{t}^2 + \dfrac{1}{4a^2}\right)^2}, \quad \max_{x,y} \upsilon(x, y) = \frac{8}{16a^4\hat{t}^2 + 1}. \tag{45}$$

It follows that an observer located in a frame at rest will see a pulse supported on the line $z = 0$ appearing at $\hat{t} = 0$ but it soon ebbs away. Thus the configuration is not a solitary wave.

4.2. Remark on the standard N-lump solution

The standard N-lump solution is constructed by simply considering a superposition of poles with similar properties, i.e, pairs of simple poles $k_j, -\bar{k}_j, j = 1, \dots N$ of index one with associated norming constants γ_j and $\gamma_{\bar{j}} = \bar{\gamma}_j$. Asymptotically, such a solution behaves simply as a superposition of modes given by (41). A more interesting case from a dynamical perspective is considered next.

4.3. Nonstandard two-lump solution

Here we construct a different two-lump solution which has interesting dynamics. Let $Z \equiv X - \beta Y$ and $-g \equiv \frac{f_1^2}{2} + it + \frac{y}{k_1^3} - \delta$ (see Equation (23)). Then this nonstandard two-lump solution is given by $|u|^2 = 1 - \partial_{yx} \log\tau$ where the τ function is

$$\tau = |g|^2 + \frac{1}{4a^2}\left[\left(Z + \frac{1}{2a}\right)^2 + \rho^2 Y^2\right] + \frac{1}{16a^4}. \tag{46}$$

and $k_1 \equiv a + ib, g = g_R + ig_I, \varepsilon = \frac{a^2-3b^2}{a^2+b^2}\frac{a}{b}$. Note also that, in terms of the coordinates coordinates defined by Equations (39) we have

$$g_R = \frac{1}{2}Z^2 - \frac{2a^2b^2}{(a^2 + b^2)^4}Y^2 + \varepsilon\left(\frac{bY}{(a^2 + b^2)^2} - t + \frac{\gamma_I}{2}\right) \tag{47}$$

$$g_I = \frac{2ab}{(a^2+b^2)^2} ZY + \frac{b(3a^2-b^2)}{(a^2+b^2)^3} Y - \frac{4a^2}{a^2+b^2} t. \tag{48}$$

From a spectral perspective this solution is associated to a meromorphic eigenfunction with a simple and a *double pole*; concretely, it is uniquely associated to the following eigenfunction

$$\mu = 1 + \frac{\phi_1}{k-k_1} + \frac{\phi_{\bar{1}}}{k+\bar{k}_1} + \frac{\psi_{\bar{1}}}{(k+\bar{k}_1)^2}. \tag{49}$$

provided that both poles have index 2. Thus, the LC at the simple pole k_1 satisfy (23) while the coefficients of the double pole satisfy (30), (32). The proof of this fact is given in Appendix.

In the sequel we concentrate in the study of the dynamical properties of this field. To simplify calculations we take $a = b$. A plot of the solution (see Figures 1, 2 below) shows that it corresponds to a two-humped configuration. A closer inspection shows *for each hump* a structure similar to that of the standard lump with several maxima and minima. However the dynamical behavior of this configuration does not correspond—not even asymptotically—to superposition of basic one-lumps. We next show that lump dynamics *consists of a constant velocity "center of mass" motion upon which a slower, lump dependent motion is superseded*. Hence, regardless of whether or not scattering happens, lump dynamics can be thought of as arising from a certain interaction force between them inversely proportional to the cube of the distance. Actually, lumps experience a head-on collision upon which they scatter and get rotated a certain scattering angle. In the sequel we substantiate the above claims.

Let $p^{\pm\infty} \equiv (Z^{\pm\infty}(t),\ Y^{\pm\infty}(t))$ be the lumps positions as $t \to \pm\infty$ with respect to the reference system $(Z,\ Y)$ where $Z \equiv X - \beta Y$. We seek to

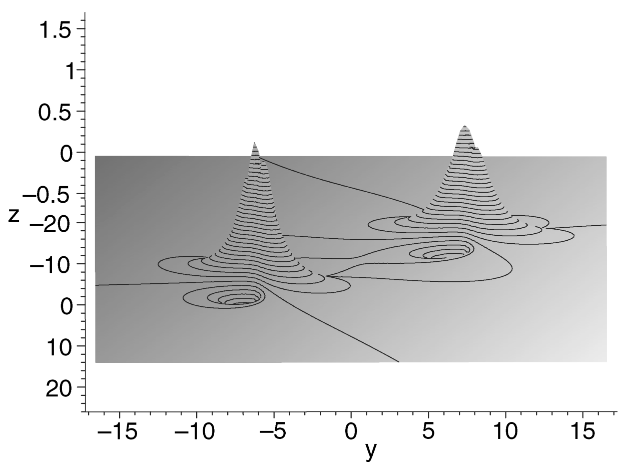

Figure 1. Nonstandard lumps at $t = -10$.

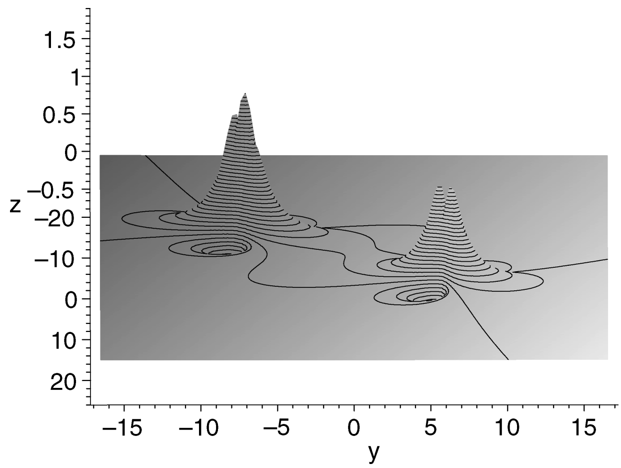

Figure 2. Nonstandard lumps at $t = 10$.

satisfy $\tau(Z^{\pm\infty}, Y^{\pm\infty}, t) = O(t)$ to dominant orders. Note that calculation shows that in this case we also have $\tau_y, \tau_x = O(t)$ and $u(Z^{\pm\infty}(t), Y^{\pm\infty}(t), t) = O(1)$—thereby defining the lump positions. We find that $p^{\pm\infty}$ can be expanded as

$$Z^{\pm\infty} = \zeta^{\pm}\sqrt{|t|} + z_0^{\pm\infty} + O(|t|^{-1/2}), \;\; Y^{\pm} = \iota^{\pm}\sqrt{|t|} + \iota_0^{\pm} + O(|t|^{-1/2}) \quad (50)$$

$$\text{where } \zeta^{\pm} = (\sqrt{5} \mp 1)^{1/2}, \iota^{\pm} \equiv \pm 4a^2/\zeta^{\pm},$$

$$\iota_0^{\pm} = -\frac{4a^2(4 \mp \sqrt{5}) + \sqrt{5} \pm 1}{4a(3\sqrt{5} - 5)}, z_0^{\pm} = \frac{-4a^2 \pm \sqrt{5} - 1}{4a^3(5 \mp \sqrt{5})}. \quad (51)$$

Also, if $p^{\pm\infty}$ is one of the lump positions then so it is $-p^{\pm\infty}$.

In the coordinates $X = x - 2at - \tilde{\gamma}, Y = y + 4a^3 t + \frac{\gamma_I}{\rho}$ the asymptotic trajectory is given by the line

$$Y^{\pm\infty} - Y_0^{\pm\infty} = \frac{4a^2}{\sqrt{5} \mp 1}\big(X^{\pm\infty} - X_0^{\pm\infty}\big). \quad (52)$$

For long times the solution has two separate lumps $p^{\pm\infty}$ and $-p^{\pm\infty}$, which, as seen in the Galilean frame (39), move along a *common* straight line, cf. Equation (52). They approach each other and after a frontal collision scatter off. The scattering angle Ω follows from Equations (50) as

$$\cos\Omega = 2\frac{1 - 4a^4}{\sqrt{(3 + 8a^4)^2 - 5}}. \quad (53)$$

Since the two lumps are indistinguishable upon scattering, we take the convention that as a increases from 0 to $a = \frac{1}{\sqrt{2}}$ so it does Ω varying from 0 to a maximum scattering angle $\pi/2$, upon which the angle decreases asymptotically to zero for large values of the parameter.

We have depicted above this situation and show the lump positions before and after scattering corresponding to $a = b = 1/2$ (see Figures (1, 2)). It is interesting to point out that this scattering process presents novel features as compared to what happens with lumps in DSII, where the scattering angle is necessarily normal or KPI where the angle is a monotone function of the real part of the spectral parameter.

In the original "laboratory frame," the lump's dynamics is a composition of two motions: an uniform motion, common to the whole structure, i.e., to both lumps and a second, slower one, that behaves as $|t|^{1/2}$ and depends on the individual lump. Notice that the "center of mass" motion can be eliminated away by proper choice of Galilean frame. By contrast, the latter although slower, is the only nontrivial physical effect and can be thought of as arising due to a force between particles which makes the particles to scatter off. In the "laboratory frame" scattering is different as viewed from the "center of mass" one. Indeed lumps move along the parabolas given by

$$(\sqrt{5} \mp 1)(y + 2a^2x)^2 + (3 \mp \sqrt{5})^2 ay - 2a(3 \mp \sqrt{5})(y + 2a^2x) = 0. \quad (54)$$

Thus, for long times both lumps move along almost parallel paths, that can be approximated by straight lines.

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Appendix

Here we show how to obtain solution (46). Assume that the eigenfunction μ is given by (49) where both poles $k_1, -\bar{k}_1$ have index 2. It follows from this representation and series expansion that the principal LCs $\nu_{1,0} \equiv \nu_{k_1,0}, \nu_{1,1} \equiv \nu_{k_1,1}, \nu_{\bar{1},0} \equiv \nu_{-\bar{k}_1,0}$ are given by (see Equation (9) and discussion below)

$$\nu_{1,0} = 1 + \frac{\phi_{\bar{1}}}{k_1 + \bar{k}_1} + \frac{\psi_{\bar{1}}}{(k_1 + \bar{k}_1)^2}, -\nu_{1,1} = \frac{\phi_{\bar{1}}}{(k_1 + \bar{k}_1)^2} + \frac{2\psi_{\bar{1}}}{(k_1 + \bar{k}_1)^3},$$
$$\text{and } \nu_{\bar{1},0} = 1 - \frac{\phi_1}{k_1 + \bar{k}_1}. \quad (A.1)$$

Furthermore, Equations (30) and (23) imply, respectively, that

$$\phi_{\bar{1}} = f_{\bar{1}}\psi_{\bar{1}},\, \nu_{\bar{1},0} = h_{\bar{1}}\psi_{\bar{1}} \text{ and } \nu_{1,1} = f_1\nu_{1,0} + g_1\phi_1 \tag{A.2}$$

where we recall (32) that $f_1 \equiv -x - 2ik_1t + \frac{y}{k_1^2} + \gamma,\; f_{\bar{1}} = -x + 2i\bar{k}_1t + \frac{y}{\bar{k}_1^2} + \gamma_{\bar{1}}$,

$$h_{\bar{1}} \equiv h(-\bar{k}_1) = \frac{f_{\bar{1}}^2}{2} - it + \frac{y}{\bar{k}_1^3} + \delta_{\bar{1}}, \quad g_1 \equiv -\left(\frac{f_1^2}{2} + it + \frac{y}{k_1^3} - \delta_1\right) \tag{A.3}$$

and we take $\gamma_{\bar{1}} = \bar{\gamma}_1, \delta_{\bar{1}} = -\bar{\delta}_1$. Notice that $f_{\bar{1}} = \bar{f}_1,\; h_{\bar{1}} = -\bar{g}_1$. With $\alpha \equiv \frac{1}{2a}$ we find upon insertion in Equation (A.1) the following system of equations for the unknowns ϕ_1 and $\psi_{\bar{1}}$:

$$g_1\phi_1 + \left(\alpha^2(f_1 + f_{\bar{1}}) + 2\alpha^3 + \alpha f_1 f_{\bar{1}}\right)\psi_{\bar{1}} = -f_1, \quad \alpha\phi_1 - \bar{g}_1\psi_{\bar{1}} = 1. \tag{A.4}$$

We solve this by Cramer's rule. Reminding that $\phi_{\bar{1}} = f_{\bar{1}}\psi_{\bar{1}}$ substitution into Equation (6) gives

$$|u|^2 = 1 + \partial_y(\phi_1 + \phi_{\bar{1}}) = 1 - \partial_{yx}\log\tau \tag{A.5}$$

where τ is the determinant of the associated matrix which coincides with (46).

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