From constants of motion to superposition rules for Lie–Hamilton systems

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Abstract

A *Lie system* is a nonautonomous system of first-order differential equations possessing a *superposition rule*, i.e. a map expressing its general solution in terms of a generic finite family of particular solutions and some constants. Lie–Hamilton systems form a subclass of Lie systems whose dynamics is governed by a curve in a finite-dimensional real Lie algebra of functions on a Poisson manifold. It is shown that Lie–Hamilton systems are naturally endowed with a Poisson coalgebra structure. This allows us to devise methods to derive in an algebraic way their constants of motion and superposition rules. We illustrate our methods by studying Kummer–Schwarz equations, Riccati equations, Ermakov systems and Smorodinsky–Winternitz systems with time-dependent frequency.

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1 Introduction

The importance of geometric methods for the study of differential equations is unquestionable. They have led to the description of many interesting properties of differential equations, which in turn have been applied in remarkable mathematical and physical problems [1]-[3]. Among geometric methods, we here focus on those of the theory of Lie systems [4]-[9].

Lie systems have lately been analysed thoroughly (see [9] and references therein). This has given rise to new techniques that have been employed to study relevant differential equations occurring in physics [10]-[12], mathematics [13], control theory [14], economy [15], etc.

The celebrated *Lie–Scheffers Theorem* [4, 7] states that a Lie system amounts to a t-dependent vector field taking values in a finite-dimensional real Lie algebra of vector fields, a so-called *Vessiot–Guldberg Lie algebra* of the system [4, 7]. In this work we concentrate on analysing a subclass of Lie systems on Poisson manifolds, the referred to as *Lie–Hamilton systems* [16], that admit a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to a Poisson structure.

As an example of Lie–Hamilton systems, consider the first-order system of differential equations on $\mathcal{O} = \{(x, p) \in \mathrm{T}^*\mathbb{R} \mid p < 0\}$ of the form

$$\left\{ \begin{array}{ll} \frac{\mathrm{d}x}{\mathrm{d}t} & = \frac{1}{\sqrt{-p}} - a_0(t) - a_1(t)x - a_2(t)x^2, \\ \frac{\mathrm{d}p}{\mathrm{d}t} & = p(a_1(t) + 2a_2(t)x), \end{array} \right. \tag{1}$$

with $a_0(t), a_1(t), a_2(t)$ being arbitrary functions, occurring in the study of some second-order Riccati equations [16, 17]. This system is a Lie–Hamilton system as it describes the integral curves of the t-dependent vector field

$$X_t = X_1 - a_0(t)X_2 - a_1(t)X_3 - a_2(t)X_4,$$

satisfying that the vector fields $\{X_t\}_{t\in\mathbb{R}}$ are contained in the five-dimensional real Lie algebra of vector fields on $\mathcal{O}$ spanned by

$$X_1 = \frac{1}{\sqrt{-p}}\frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x\frac{\partial}{\partial x} - p\frac{\partial}{\partial p}, \quad X_4 = x^2\frac{\partial}{\partial x} - 2xp\frac{\partial}{\partial p},$$
$$X_5 = \frac{x}{\sqrt{-p}}\frac{\partial}{\partial x} + 2\sqrt{-p}\frac{\partial}{\partial p}, \tag{2}$$

which are additionally Hamiltonian vector fields relative to the Poisson bivector $\Lambda = \partial/\partial x \wedge \partial/\partial p$ on $\mathcal{O}$. Indeed, they admit the Hamiltonian functions

$$h_1 = -2\sqrt{-p}, \quad h_2 = p, \quad h_3 = xp, \quad h_4 = x^2p, \quad h_5 = -2x\sqrt{-p}, \tag{3}$$

which span along with $h_6 = 1$ a six-dimensional real Lie algebra of functions with respect to the Poisson bracket induced by Λ (see [17] for details).

Lie–Hamilton systems can be employed to investigate remarkable dynamical sytems and enjoy a plethora of geometric properties [16, 18, 19, 20]. For example, the Vessiot–Guldberg Lie algebras of Hamiltonian vector fields associated to these systems give rise to t-dependent Hamiltonians that can be understood as curves in finite-dimensional real Lie algebras of functions, the hereafter called *Lie–Hamilton algebras* [16]. This property has recently been employed to study t-independent constants of motion of Lie–Hamilton systems [16]. Note that t generally stands for the time when referring to Lie–Hamilton systems describing a physical system.

In this work, we first employ Lie–Hamilton algebras so as to analyse the constants of motion for Lie–Hamilton systems. This provides some generalisations of the results given in [16] about t-independent constants of motion and in [20] about t-dependent constants of motion for Lie–Hamilton systems. In addition, we demonstrate that several types of constants of motion of Lie–Hamilton systems form a Poisson algebra, and we devise algebraic procedures to derive them or, at least, to simplify their calculation.

We propose the use of *Poisson coalgebra* techniques to obtain superposition rules for Lie–Hamilton systems in an algebraic manner. We recall that Poisson coalgebras are Poisson algebras endowed with a Poisson algebra homomorphism called the coproduct map [21]. We show that the coproduct map applied onto a Casimir function of an appropriate Poisson coalgebra naturally related to a Lie–Hamilton system X gives rise to t-independent constants of motion that can be employed to study the superposition rules and the N-dimensional generalisations of X.

Since our procedures are algebraic, they are much simpler than standard methods [6, 7], which require the integration of systems of partial/ordinary differential equations. As an application, we derive constants of motion and superposition rules for some Lie–Hamilton systems of interest: Kummer–Schwarz equations in Hamiltonian form [16], systems of Riccati equations [7], Smorodinsky–Winternitz systems with time-dependent frequency [22, 23], and some classical mechanical systems [18, 24, 25].

The structure of the paper goes as follows. We describe the conventions and the most fundamental notions to be used throughout our paper in Section 2. In Section 3, we introduce the fundamental features of Lie–Hamilton systems. In Section 4, we analyse the algebraic structure of general constants of motion for Lie–Hamilton systems. Subsequently, we focus on the study of several relevant particular types of such constants in Section 5. This involves the description of some new methods for their calculation. In Section 6, Poisson coalgebras are shown to provide new simpler methods to derive constants of motion and superposition rules for Lie–Hamilton systems. Several examples are analysed in Section 7. Finally, we summarise our results and plan of future research in Section 8.

2 Preliminaries

In this section we survey t-dependent vector fields [9], Poisson algebras [16] and Poisson coalgebra structures [21, 26]. For simplicity, we generally assume functions and geometric structures to be real, smooth, and globally defined. This permits us to omit minor technical problems so as to highlight the main aspects of our results.

A Lie algebra is a pair $(V, [\cdot, \cdot])$, where V stands for a real linear space equipped with a Lie bracket $[\cdot\,, \cdot] : V \times V \to V$. We define $\mathrm{Lie}(\mathcal{B}, V, [\cdot, \cdot])$ to be the smallest Lie subalgebra of $(V, [\cdot, \cdot])$ containing $\mathcal{B}$. When its meaning is clear, we write V and $\mathrm{Lie}(\mathcal{B})$ instead of $(V, [\cdot, \cdot])$ and $\mathrm{Lie}(\mathcal{B}, V, [\cdot, \cdot])$, respectively,

A *t-dependent vector field* on a manifold N is a map $X : (t, x) \in \mathbb{R} \times N \mapsto X(t, x) \in \mathrm{T}N$ such that $\tau_N \circ X = \pi_2$, with $\pi_2 : (t, x) \in \mathbb{R} \times N \mapsto x \in N$ and $\tau_N : \mathrm{T}N \to N$ being the projection associated to the tangent bundle $\mathrm{T}N$. This amounts to saying that X is a t-parametrized family of standard vector fields $\{X_t\}_{t\in\mathbb{R}}$, with $X_t : x \in N \mapsto X(t, x) \in \mathrm{T}N$. We call *minimal Lie algebra* of X the smallest real Lie algebra of vector fields, V^X, containing

$\{X_t\}_{t\in\mathbb{R}}$, namely $V^X = \text{Lie}(\{X_t\}_{t\in\mathbb{R}})$. We can also consider each X as a vector field on $\mathbb{R}\times N$ by defining $(Xf)(t,x) = (X_t f_t)(x)$, where $f \in C^\infty(\mathbb{R}\times N)$ and $f_t \in C^\infty(N)$ stands for $f_t : x \in N \mapsto f(t,x) \in \mathbb{R}$.

We call *integral curves* of X the integral curves of its *suspension*, i.e. the vector field $\bar{X}(t,x) = \partial/\partial t + X(t,x)$ on $\mathbb{R}\times N$ [9, 27, 28]. Every integral curve $\gamma : t \in \mathbb{R} \mapsto (t, x(t)) \in \mathbb{R}\times N$ of X satisfies the *associated system* to X, i.e.

$$\frac{\mathrm{d}(\pi_2\circ\gamma)}{\mathrm{d}t}(t) = (X\circ\gamma)(t). \tag{4}$$

Conversely, there exists a unique X whose integral curves of the form $(t, x(t))$ describe the solutions of a system of first-order systems in normal form. This establishes a bijection between t-dependent vector fields and such systems, which justifies to use X to designate both X and (4).

A *Poisson algebra* is a triple $(A, \star, \{\cdot,\cdot\})$ consisting of an $\mathbb{R}$-linear space A along with a product $\star : A\times A \to A$ so that $(A,\star)$ becomes an associative $\mathbb{R}$-algebra, and a Lie bracket $\{\cdot,\cdot\}$ on A, the so-called *Poisson bracket*, such that $(A,\{\cdot,\cdot\})$ is a (possibly infinite-dimensional) Lie algebra and

$$\{b\star c, a\} = b\star\{c,a\} + \{b,a\}\star c, \qquad \forall a,b,c\in A.$$

The above compatibility condition, the called *Leibnitz rule*, amounts to saying that the Poisson bracket is a derivation of $(A,\star)$ on each factor. Contrarily to the usual convention, the product $\star$ of our definition of Poisson algebras may be non-commutative, which simplifies posterior definitions. We call *Casimir element* of A an $a\in A$ such that $\{c,a\} = 0$ for all $c\in A$. The set $\text{Cas}(A)$ of Casimir elements of A is an ideal of $(A,\{\cdot,\cdot\})$. For simplicity, we will sometimes write ab for the $a\star b$. A *Poisson algebra morphism* is a morphism $\mathcal{T} : (A, \star_A, \{\cdot,\cdot\}_A) \to (B, \star_B, \{\cdot,\cdot\}_B)$ of $\mathbb{R}$-algebras $\mathcal{T} : (A,\star_A) \to (B,\star_B)$ that also satisfies that $\mathcal{T}(\{a,b\}_A) = \{\mathcal{T}(a), \mathcal{T}(b)\}_B$ for every $a,b\in A$.

One of the types of Poisson algebras to be used in this paper are symmetric and universal algebras. Let us describe their main characteristics. Given a finite-dimensional real Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$, its universal algebra, $U_\mathfrak{g}$, is obtained from the quotient $T_\mathfrak{g}/\mathcal{R}$ of the tensor algebra $(T_\mathfrak{g}, \otimes)$ of $\mathfrak{g}$ by the bilateral ideal $\mathcal{R}$ spanned by the elements $v\otimes w - w\otimes v - [v,w]$, with $v,w\in\mathfrak{g}$. Given the quotient map $\pi : T_\mathfrak{g} \to U_\mathfrak{g}$, the space $U_\mathfrak{g}$ becomes an $\mathbb{R}$-algebra $(U_\mathfrak{g}, \widetilde{\otimes})$ when endowed with the product $\widetilde{\otimes} : U_\mathfrak{g}\times U_\mathfrak{g} \to U_\mathfrak{g}$ given by $\pi(P)\,\widetilde{\otimes}\,\pi(Q) \equiv \pi(P\otimes Q)$, for every $P,Q\in T_\mathfrak{g}$. The Lie bracket on $\mathfrak{g}$ can be extended to a Lie bracket $\{\cdot,\cdot\}_{U_\mathfrak{g}}$ on $U_\mathfrak{g}$ by imposing it to be a derivation of $(U_\mathfrak{g}, \widetilde{\otimes})$ on each factor. This turns $U_\mathfrak{g}$ into a Poisson algebra $(U_\mathfrak{g}, \widetilde{\otimes}, \{\cdot,\cdot\}_{U_\mathfrak{g}})$ [29]. The elements of its Casimir subalgebra are henceforth dubbed as *Casimir elements* of $\mathfrak{g}$ [30].

If we set $\mathcal{R}$ to be the bilateral ideal spanned by the elements $v\otimes w - w\otimes v$ in the above procedure, we obtain a new commutative Poisson algebra $S_\mathfrak{g}$ called *symmetric algebra* of $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$. The elements of $S_\mathfrak{g}$ are polynomials on the elements of $\mathfrak{g}$. Via the isomorphism $\mathfrak{g} \simeq (\mathfrak{g}^*)^*$, they can naturally be understood as polynomial functions on $\mathfrak{g}^*$ [29, 31]. The Casimir elements of this Poisson algebra are called *Casimir functions* of $\mathfrak{g}$.

The Poisson algebras $U_{\mathfrak{g}}$ and $S_{\mathfrak{g}}$ are related by the *symmetrizer map* [30, 31, 32], i.e. the linear isomorphism $\lambda : S_{\mathfrak{g}} \to U_{\mathfrak{g}}$ of the form

$$\lambda(v_{i_1}) = \pi(v_{i_1}), \quad \lambda(v_{i_1} v_{i_2} \dots v_{i_l}) = \frac{1}{l!} \sum_{s \in \Pi_l} \lambda(v_{s(i_1)}) \,\widetilde{\otimes} \dots \widetilde{\otimes}\, \lambda(v_{s(i_l)}), \tag{5}$$

for all $v_{i_1}, \dots, v_{i_l} \in \mathfrak{g}$ and with Π_l being the set of permutations of l elements. Moreover,

$$\lambda^{-1}\left(\{v, P\}_{U_{\mathfrak{g}}}\right) = \{v, \lambda^{-1}(P)\}_{S_{\mathfrak{g}}}, \qquad \forall P \in U_{\mathfrak{g}}, \quad \forall v \in \mathfrak{g}. \tag{6}$$

So, λ^{-1} maps the Casimir elements of $\mathfrak{g}$ into Casimir elements of $S_{\mathfrak{g}}$.

If $(A, \star_A, \{\cdot,\cdot\}_A)$ and $(B, \star_B, \{\cdot,\cdot\}_B)$ are Poisson algebras and $\star_A$, $\star_B$ are commutative, then $A \otimes B$ becomes a Poisson algebra $(A \otimes B, \star_{A\otimes B}, \{\cdot,\cdot\}_{A\otimes B})$ by defining

$$\begin{aligned} (a \otimes b) \star_{A\otimes B} (c \otimes d) &= (a \star_A c) \otimes (b \star_B d), \qquad \forall a, c \in A, \quad \forall b, d \in B, \\ \{a \otimes b, c \otimes d\}_{A\otimes B} &= \{a, c\}_A \otimes b \star_B d + a \star_A c \otimes \{b, d\}_B. \end{aligned}$$

Similarly, a Poisson structure on $A^{(m)} \equiv \overbrace{A \otimes \dots \otimes A}^{m-\text{times}}$ can be constructed by induction.

We say that $(A, \star_A, \{\cdot,\cdot\}_A, \Delta)$ is a *Poisson coalgebra* if $(A, \star_A, \{\cdot,\cdot\}_A)$ is a Poisson algebra and $\Delta : (A, \star_A, \{\cdot,\cdot\}_A) \to (A \otimes A, \star_{A\otimes A}, \{\cdot,\cdot\}_{A\otimes A})$, the so-called *coproduct*, is a Poisson algebra homomorphism which is *coassociative* [21], i.e. $(\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta$. Then, the m-th coproduct map $\Delta^{(m)} : A \to A^{(m)}$ can be defined recursively as follows

$$\Delta^{(m)} = (\overbrace{\mathrm{Id} \otimes \dots \otimes \mathrm{Id}}^{(m-2)-\text{times}} \otimes \Delta^{(2)}) \circ \Delta^{(m-1)}, \qquad m > 2, \tag{7}$$

and $\Delta \equiv \Delta^{(2)}$. Such an induction ensures that $\Delta^{(m)}$ is also a Poisson map.

In particular, $S_{\mathfrak{g}}$ is a Poisson coalgebra with *primitive coproduct map* given by $\Delta(v) = v \otimes 1 + 1 \otimes v$, for all $v \in \mathfrak{g} \subset S_{\mathfrak{g}}$. The coassociativity of Δ is straightforward, and its m-th generalization reads

$$\Delta^{(m)}(v) = v \otimes \overbrace{1 \otimes \dots \otimes 1}^{(m-1)-\text{times}} + 1 \otimes v \otimes \overbrace{1 \otimes \dots \otimes 1}^{(m-2)-\text{times}} + \dots + \overbrace{1 \otimes \dots \otimes 1}^{(m-1)-\text{times}} \otimes v, \quad \forall v \in \mathfrak{g} \subset S_{\mathfrak{g}}.$$

A *Poisson manifold* is a pair $(N, \{\cdot,\cdot\})$ such that $(C^\infty(N), \star, \{\cdot,\cdot\})$, where "$\star$" stands for the standard product of functions, is a Poisson algebra. We call $\{\cdot,\cdot\}$ the *Poisson structure* of the Poisson manifold. For instance, the dual space $\mathfrak{g}^*$ of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ can be endowed with the *Lie–Poisson structure* [33], namely

$$\{f, g\}(\theta) = \langle [df_\theta, dg_\theta]_{\mathfrak{g}}, \theta \rangle, \qquad f, g \in C^\infty(\mathfrak{g}^*), \qquad \theta \in \mathfrak{g}^*.$$

As every $\{\cdot, f\}$, with $f \in C^\infty(N)$, is a derivation on $(C^\infty(N), \star)$, there exists a unique vector field X_f on N, the referred to as *Hamiltonian vector field* associated with f, such that $X_f g = \{g, f\}$ for all $g \in C^\infty(N)$. The Jacobi identity for the Poisson structure then entails $X_{\{f,g\}} = -[X_f, X_g]$, $\forall f, g \in C^\infty(N)$. Hence, the mapping $f \mapsto X_f$ is a Lie algebra anti-homomorphism from $(C^\infty(N), \{\cdot,\cdot\})$ to $(\Gamma(\mathrm{T}N), [\cdot,\cdot])$, where $\Gamma(\mathrm{T}N)$ is the space of sections of the tangent bundle to N.

As every Poisson structure is a derivation in each entry, it determines a unique bivector field $\Lambda \in \Gamma(\bigwedge^2 \mathrm{T}N)$ such that

$$\{f, g\} = \Lambda(df, dg), \qquad \forall f, g \in C^\infty(N). \tag{8}$$

We call Λ the *Poisson bivector* of the Poisson manifold $(N, \{\cdot,\cdot\})$. In view of the Jacobi identity for $\{\cdot,\cdot\}$, it follows that $[\Lambda, \Lambda]_{\mathrm{SN}} = 0$, with $[\cdot,\cdot]_{\mathrm{SN}}$ being the *Schouten–Nijenhuis Lie bracket* [33]. Conversely, every bivector field Λ on N satisfying this condition gives rise to a Poisson structure by formula (8). This justifies to refer to Poisson manifolds as $(N, \{\cdot,\cdot\})$ or (N, Λ) indistinctly.

Observe that (N, Λ) induces a unique bundle morphism $\widehat{\Lambda} : \mathrm{T}^*N \to \mathrm{T}N$ satisfying $\theta'(\widehat{\Lambda}(\theta)) = \Lambda(\theta, \theta')$ for every $\theta, \theta' \in \Gamma(T^*N)$. So, $X_f = -\widehat{\Lambda}(df)$, for any $f \in C^\infty(N)$.

3 Lie systems and Lie–Hamilton systems

Recall that Lie systems are systems of first-order differential equations admitting a superposition rule (see [7, 9] for details).

Definition 1. A *superposition rule* for a system X on N is a function $\Phi : N^m \times N \to N$ of the form $x = \Phi(x_{(1)}, \ldots, x_{(m)}; k)$ allowing us to write the general solution $x(t)$ of X as $x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t); k)$, where $x_{(1)}(t), \ldots, x_{(m)}(t)$ is a generic family of particular solutions and k is a point of N.

The conditions ensuring that a system X possesses a superposition rule are described by the *Lie–Scheffers Theorem* [4, 7, 9].

Theorem 2. *A system X on N admits a superposition rule if and only if $X_t = \sum_{\alpha=1}^r b_\alpha(t) X_\alpha$ for a family $b_1(t), \ldots, b_r(t)$ of t-dependent functions and a set of vector fields $X_1, \ldots, X_r$ on N spanning an r-dimensional real Lie algebra, a so-called Vessiot–Guldberg Lie algebra associated to X [34]. In other words, X is a Lie system if and only if V^X is finite-dimensional.*

In order to illustrate the above concepts, let us consider a *Riccati equation*, i.e.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a_0(t) + a_1(t)x + a_2(t)x^2, \tag{9}$$

where $a_0(t), a_1(t), a_2(t)$ are arbitrary functions. Observe that (9) is the system associated to the t-dependent vector field $X = a_0(t)X_1 + a_1(t)X_2 + a_2(t)X_3$, where

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x}, \qquad X_3 = x^2\frac{\partial}{\partial x}$$

span a Vessiot–Guldberg Lie algebra V for (9) isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ [6]. The Lie–Scheffers Theorem shows that Riccati must admit a superposition rule. Indeed, the general solution of (9) can be brought into the form

$$x(t) = \frac{x_1(t)(x_3(t)-x_2(t)) + kx_3(t)(x_1(t)-x_2(t))}{(x_3(t)-x_2(t)) + k(x_1(t)-x_2(t))},$$

where $x_{(1)}(t), x_{(2)}(t), x_{(3)}(t)$ are three different particular solutions and k is an arbitrary constant. So, Riccati equations possess a superposition rule $\Phi : (x_{(1)}, x_{(2)}, x_{(3)}; k) \in \mathbb{R}^3 \times \mathbb{R} \mapsto x \in \mathbb{R}$ given by

$$x = \frac{x_1(x_3-x_2) + kx_3(x_1-x_2)}{(x_3-x_2) + k(x_1-x_2)}$$

enabling us to write their general solutions as $x(t) = \Phi(x_{(1)}(t), x_{(2)}(t), x_{(3)}(t); k)$.

There exist several ways to derive superposition rules for Lie systems [7, 35]. The method employed in this work (see [13, p. 4] and [36] for a detailed description and examples) is based on the so-called *diagonal prolongations*.

Definition 3. Given a t-dependent vector field X on N, its *diagonal prolongation* to N^{m+1} is the unique t-dependent vector field $\widetilde{X}$ on N^{m+1} such that:

1. Each vector field $\widetilde{X}_t$ projects onto X_t via $\Pi : (x_{(0)}, \ldots, x_{(m)}) \in N^{m+1} \mapsto x_{(0)} \in N$.
2. $\widetilde{X}$ is invariant under permutations of variables $x_{(i)} \leftrightarrow x_{(j)}$, with $i, j = 0, \ldots, m$.

Roughly speaking, we can derive superposition rules by obtaining a certain set of constants of motion for an appropriate diagonal prolongation of a Lie system. In the literature, this is usually performed by solving a system of PDEs by the method of characteristics (cf. [16]), which can be quite laborious [36]. As seen in this work, this task can be simplified in the case of *Lie–Hamilton systems* through our methods.

Definition 4. A system X on N is said to be a *Lie–Hamilton system* if N can be endowed with a Poisson bivector Λ in such a way that V^X becomes a finite-dimensional real Lie algebra of Hamiltonian vector fields relative to Λ.

Lie–Hamilton systems enjoy relevant features, e.g. (1) admits a t-dependent Hamiltonian

$$h(t) = h_1 - a_0(t)h_2 - a_1(t)h_3 - a_2(t)h_4. \tag{10}$$

The functions $h_1, \ldots, h_5$ and $h_6 = 1$ given by (3) span a six-dimensional real Lie algebra $(\mathfrak{h}_6, \{\cdot,\cdot\}_\Lambda)$. It is also relevant that $\mathfrak{h}_6 \simeq \mathfrak{sl}(2,\mathbb{R}) \oplus_s \mathfrak{h}_3$, where $\mathfrak{sl}(2,\mathbb{R}) \simeq \langle h_2, h_3, h_4\rangle$, $\mathfrak{h}_3 \simeq \langle h_1, h_5, h_6\rangle$ is the radical of $\mathfrak{h}_6$, which is isomorphic to the Heisenberg–Weyl Lie algebra, and $\oplus_s$ stands for a semidirect sum. Additionally, $\mathfrak{h}_6$ is also isomorphic to the two-photon Lie algebra and the (1+1)-dimensional centrally extended Schrödinger Lie algebra (cf. [37, 38]).

The t-dependent Hamiltonians of the form (10) are called *Lie–Hamiltonian structures*. More specifically, we have the following definition.

Definition 5. A *Lie–Hamiltonian structure* is a triple (N, Λ, h), where (N, Λ) stands for a Poisson manifold and $h : (t, x) \in \mathbb{R} \times N \mapsto h_t(x) = h(t, x) \in N$ is such that the space $(\mathcal{H}_\Lambda \equiv \mathrm{Lie}(\{h_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_\Lambda), \{\cdot,\cdot\}_\Lambda)$ is a finite-dimensional real Lie algebra.

We now describe some features of Lie–Hamilton systems (see [16] for details).

Theorem 6. *A system X on N is a Lie–Hamilton system if and only if there exists a Lie–Hamiltonian structure (N, Λ, h) such that $X_t = -\widehat{\Lambda}(dh_t)$ for every $t \in \mathbb{R}$. In this case, we call $(\mathcal{H}_\Lambda, \{\cdot,\cdot\}_\Lambda)$ a Lie–Hamilton algebra for X.*

Proposition 7. *Given a Lie–Hamilton system X on N admitting a Lie–Hamiltonian structure (N, Λ, h), a t-independent function is a constant of motion for X if and only if it Poisson commutes with the elements of $\mathcal{H}_\Lambda$. The family $\mathcal{I}^X$ of t-independent constants of motion of X form a Poisson algebra $(\mathcal{I}^X, \cdot, \{\cdot,\cdot\}_\Lambda)$.*

4 On constants of motion for Lie–Hamilton systems

The aim of this section is to present an analysis of the algebraic properties of the constants of motion for Lie–Hamilton systems. More specifically, we prove that a Lie–Hamiltonian structure (N, Λ, h) for a Lie–Hamilton system X induces a Poisson bivector on $\mathbb{R} \times N$. This allows us to endow the space of constants of motion for X with a Poisson algebra structure, which can be used to produce new constants of motion from known ones. Our achievements extend to general constants of motion the results derived for t-independent constants of motion for Lie–Hamilton systems in [16].

Given a system X on N, a constant of motion for X is a first-integral $f \in C^\infty(\mathbb{R} \times N)$ of the autonomisation $\bar{X}$ of X, namely

$$\frac{\partial f}{\partial t} + Xf = \bar{X}f = 0, \tag{11}$$

where X is understood as a vector field on $\mathbb{R} \times N$. Using this, we can straightforwardly prove the following proposition.

Proposition 8. *The space $\bar{\mathcal{I}}^X$ of t-dependent constants of motion for a system X forms an $\mathbb{R}$-algebra $(\bar{\mathcal{I}}^X, \cdot)$.*

To generalise the second statement of Proposition 7 to t-dependent constants of motion, we endow $\mathbb{R} \times N$ with a Poisson structure that makes $\bar{\mathcal{I}}^X$ into a Poisson algebra.

Lemma 9. *Every Poisson manifold (N, Λ) induces a Poisson manifold $(\mathbb{R} \times N, \bar{\Lambda})$ with Poisson structure*

$$\{f, g\}_{\bar{\Lambda}}(t, x) \equiv \{f_t, g_t\}_\Lambda(x), \qquad (t, x) \in \mathbb{R} \times N. \tag{12}$$

Definition 10. Given a Poisson manifold (N, Λ), the associated Poisson manifold $(\mathbb{R} \times N, \bar{\Lambda})$ is called the *autonomisation* of (N, Λ). Likewise, the Poisson bivector $\bar{\Lambda}$ is called the *autonomisation* of Λ.

The following lemma allows us to prove that $(\bar{\mathcal{I}}^X, \cdot, \{\cdot,\cdot\}_{\bar{\Lambda}})$ is a Poisson algebra.

Lemma 11. *Let (N,Λ) be a Poisson manifold and X be a Hamiltonian vector field on N relative to Λ. Then, $\mathcal{L}_{\bar{X}}\bar{\Lambda} = 0$.*

Proof. Given a coordinate system $\{x_1,\ldots,x_n\}$ for N and $x_0 \equiv t$ in $\mathbb{R}$, we can naturally define a coordinate system $\{x_0,x_1,\ldots,x_n\}$ on $\mathbb{R}\times N$. Since $(x_0)_t = t$ is constant as a function on N, then $\bar{\Lambda}(dx_0, df) = \{(x_0)_t, f_t\}_\Lambda = 0$ for every $f \in C^\infty(\mathbb{R}\times N)$. Additionally, $(x_i)_t = x_i$ for $i = 1,\ldots,n$. Hence, we have that $\bar{\Lambda}(\mathrm{d}x_i, \mathrm{d}x_j) = \{x_i, x_j\}_{\bar{\Lambda}}$ is a x_0-independent function for $i,j = 0,\ldots,n$. So,

$$(\mathcal{L}_{\bar{X}}\bar{\Lambda})(t,x) = \left[\mathcal{L}_{\frac{\partial}{\partial x_0}+X}\left(\sum_{i<j=1}^{n}\{x_i,x_j\}_\Lambda \frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}\right)\right](x) = (\mathcal{L}_X\Lambda)(x).$$

Since X is Hamiltonian, we obtain $0 = (\mathcal{L}_X\Lambda)(x) = (\mathcal{L}_{\bar{X}}\bar{\Lambda})(t,x) = 0$. □

Now, we can establish the main result of this section.

Proposition 12. *Let X be a Lie–Hamilton system on N possessing a Lie–Hamiltonian structure (N,Λ,h), then the space $(\overline{\mathcal{I}}^X, \cdot, \{\cdot,\cdot\}_{\bar{\Lambda}})$ is a Poisson algebra.*

Proof. From Proposition 8, we see that $(\bar{\mathcal{I}}^X, \cdot)$ is an $\mathbb{R}$-algebra. To demonstrate that $(\bar{\mathcal{I}}^X, \cdot, \{\cdot,\cdot\}_{\bar{\Lambda}})$ is a Poisson algebra, it remains to prove that the Poisson bracket of any two elements f, g of $\bar{\mathcal{I}}^X$ remains in it, i.e. $X\{f,g\}_{\bar{\Lambda}} = 0$. By taking into account that the vector fields $\{X_t\}_{t\in\mathbb{R}}$ are Hamiltonian relative to (N,Λ) and Lemma 11, we find that $\bar{\Lambda}$ is invariant under the autonomization of each vector field $X_{t'}$ with $t' \in \mathbb{R}$, i.e. $\mathcal{L}_{\overline{X_{t'}}}\bar{\Lambda} = 0$. Therefore,

$$\begin{aligned}\bar{X}\{f,g\}_{\bar{\Lambda}}(t',x) &= \overline{X_{t'}}\{f,g\}_{\bar{\Lambda}}(t',x) = \{\overline{X_{t'}}f,g\}_{\bar{\Lambda}}(t',x) + \{f,\overline{X_{t'}}g\}_{\bar{\Lambda}}(t',x) = \\ &= \{\bar{X}f,g\}_{\bar{\Lambda}}(t',x) + \{f,\bar{X}g\}_{\bar{\Lambda}}(t',x) = 0.\end{aligned} \tag{13}$$

That is, $\{f,g\}_{\bar{\Lambda}}$ is a t-dependent constant of motion for X. □

5 Polynomial Lie integrals for Lie–Hamilton systems

Let us formally define and investigate a remarkable class of constants of motion for Lie–Hamilton systems appearing in the literature [20, 39], the hereafter called *Lie integrals*, and a relevant generalization of them, the *polynomial Lie integrals*. We first prove that Lie integrals can be characterised by an Euler equation on a finite-dimensional real Lie algebra of functions, retrieving as a particular case a result given in [20]. Then, we show that Lie integrals form a finite-dimensional real Lie algebra and we devise several methods to determine them. Our results can easily be extended to investigate certain quantum mechanical systems [40]. Finally, we investigate polynomial Lie integrals and the relevence of Casimir functions to derive them.

Definition 13. Given a Lie–Hamilton system X on N possessing a Lie–Hamiltonian structure (N, Λ, h), a *Lie integral* of X with respect to (N, Λ, h) is a constant of motion f of X such that $\{f_t\}_{t\in\mathbb{R}} \subset \mathcal{H}_\Lambda$. In other words, given a basis $h_1, \ldots, h_r$ of the Lie algebra $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda)$, we have that $\bar{X}f = 0$ and $f_t = \sum_{\alpha=1}^{r} f_\alpha(t) h_\alpha$ for every $t \in \mathbb{R}$ and certain t-dependent functions $f_1, \ldots, f_r$.

The Lie integrals of a Lie–Hamilton system X relative to a Lie–Hamiltonian structure (N, Λ, h) are the solutions of the equation

$$0 = \bar{X}f = \frac{\partial f}{\partial t} + Xf = \frac{\partial f}{\partial t} + \{f, h\}_{\bar{\Lambda}} \Longrightarrow \frac{\partial f}{\partial t} = \{h, f\}_{\bar{\Lambda}}.$$

Since f and h can be understood as curves $t \mapsto f_t$ and $t \mapsto g_t$ within $\mathcal{H}_\Lambda$, the above equation can be rewritten as

$$\frac{\mathrm{d}f_t}{\mathrm{d}t} = \{h_t, f_t\}_\Lambda, \tag{14}$$

which can be thought of as an Euler equation on the Lie algebra $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda)$ [12]. Equations of this type quite frequently appear in the literature such as in the Lewis–Riesenfeld method and works concerning Lie–Hamilton systems [20, 39, 40].

5.1 Algebraic structure of Lie integrals

Previous to the description of methods to solve equation (14), let us prove some results about the algebraic structure of its solutions and the equation itself.

Proposition 14. *Given a Lie–Hamilton system X with a Lie–Hamiltonian structure (N, Λ, h), the space $\mathfrak{L}_h^\Lambda$ of Lie integrals relative to (N, Λ, h) gives rise to a Lie algebra $(\mathfrak{L}_h^\Lambda, \{\cdot, \cdot\}_{\bar{\Lambda}})$ isomorphic to $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda)$.*

Proof. Since the Lie integrals of X are the solutions of the system of ODEs (14) on $\mathcal{H}_\Lambda$, they span an $\mathbb{R}$-linear space of dimension $\dim \mathcal{H}_\Lambda$. In view of Proposition 12, the Poisson bracket $\{f, g\}_{\bar{\Lambda}}$ of two constants of motion f, g for X is another constant of motion. If f and g are Lie integrals, the function $\{f, g\}_{\bar{\Lambda}}$ is then a new constant of motion that can additionally be considered as a curve $t \mapsto \{f_t, g_t\}_\Lambda$ taking values in $\mathcal{H}_\Lambda$, i.e. a new Lie integral.

Consider now the linear morphism $\mathcal{E}_0 : f \in \mathfrak{L}_h^\Lambda \to f_0 \in \mathcal{H}_\Lambda$ relating every Lie integral to its value in $\mathcal{H}_\Lambda$ at $t = 0$. As every initial condition in $\mathcal{H}_\Lambda$ is related to a single solution of (14), we can relate every $v \in \mathcal{H}_\Lambda$ with a unique Lie integral f of X such that $f_0 = v$. Therefore, $\mathcal{E}_0$ is an isomorphism. Indeed, it is a Lie algebra isomorphism since $(\{f, g\}_{\bar{\Lambda}})_0 = \{f_0, g_0\}_\Lambda$ for every $f, g \in \mathfrak{L}_h^\Lambda$. □

Proposition 15. *Given a Lie–Hamilton system X on N possessing a Lie–Hamiltonian structure (N, Λ, h), then $\mathfrak{L}_h^\Lambda$ consists of t-independent constants of motion if and only if $\mathcal{H}_\Lambda$ is Abelian.*

Proof. If $(\mathcal{H}_\Lambda, \{\cdot, \cdot\}_\Lambda)$ is Abelian, then $\{f_t, h_t\}_\Lambda = 0$ and the system (14) reduces to $\mathrm{d}f_t/\mathrm{d}t = 0$, whose solutions are of the form $f_t = g \in \mathcal{H}_\Lambda$, i.e. $\mathfrak{L}_h^\Lambda = \mathcal{H}_\Lambda$. Conversely, if $\mathfrak{L}_h^\Lambda = \mathcal{H}_\Lambda$, then every $g \in \mathcal{H}_\Lambda$ is a solution of (14) and $\{g, h_t\}_\Lambda = 0$ $\forall t \in \mathbb{R}$. Hence, every $g \in \mathcal{H}_\Lambda$ commutes with the whole $\mathcal{H}_\Lambda$, which becomes Abelian. □

5.2 Reduction techniques for the determination of Lie integrals

Let us turn to the study of (14) through the methods of the theory of Lie systems. A first approach to this topic can be found in [12, 20, 41]. To start with, we prove that system (14) is a Lie system related to a Vessiot–Guldberg Lie algebra V isomorphic to $(\mathcal{H}_\Lambda/Z(\mathcal{H}_\Lambda), \{\cdot,\cdot\}_\Lambda)$.

Let $\{h_1,\ldots,h_r\}$ be a basis for $\mathcal{H}_\Lambda$, we can write $h_t = \sum_{\alpha=1}^r b_\alpha(t)h_\alpha$ for certain t-dependent functions $b_1,\ldots,b_r$. In addition, system (14) becomes

$$\frac{\mathrm{d}f_t}{\mathrm{d}t} = \sum_{\alpha=1}^r b_\alpha(t)Y_\alpha(f_t),$$

where $Y_\alpha(f_t) = \mathrm{ad}_{h_\alpha}(f_t)$, with $\alpha = 1,\ldots,r$ and $\mathrm{ad} : f \in \mathcal{H}_\Lambda \mapsto \mathrm{ad}_f(g) \equiv \{f,g\}_\Lambda \in \mathcal{H}_\Lambda$ is the adjoint representation of $\mathcal{H}_\Lambda$. The vector fields $Y_1,\ldots,Y_r$ are the fundamental vector fields of the adjoint action $\mathrm{Ad} : \mathcal{G}_\Lambda \times \mathcal{H}_\Lambda \to \mathcal{H}_\Lambda$ of a Lie group $\mathcal{G}_\Lambda$ with Lie algebra isomorphic to $\mathcal{H}_\Lambda$. Consequently, they span a finite-dimensional real Lie algebra of vector fields isomorphic to $\mathrm{Im}\,\mathrm{ad} \simeq \mathcal{H}_\Lambda/\ker\mathrm{ad} = \mathcal{H}_\Lambda/Z(\mathcal{H}_\Lambda)$, where we write $Z(\mathcal{H}_\Lambda)$ for the *center* of $(\mathcal{H}_\Lambda, \{\cdot,\cdot\}_\Lambda)$. So, (14) is a Lie system related to a Vessiot–Guldberg Lie algebra isomorphic to $(\mathcal{H}_\Lambda/Z(\mathcal{H}_\Lambda), \{\cdot,\cdot\}_\Lambda)$.

Instead of applying the standard approaches of the theory of Lie systems to (14) (see for instance [5, 7, 9, 42]), we hereafter derive a more straightforward new method based upon Lie–Hamilton systems that provides similar results.

If $Z(\mathcal{H}_\Lambda)$ is not trivial, its elements are Lie integrals of X. More generally, if $(\mathcal{H}_\Lambda, \{\cdot,\cdot\}_\Lambda)$ is solvable, $\mathcal{H}_\Lambda$ admits a flag of ideals [43], i.e. there exists a family $\mathcal{H}_1,\ldots,\mathcal{H}_{r-1}$ of ideals of $\mathcal{H}_\Lambda$ satisfying that

$$\mathcal{H}_\Lambda \equiv \mathcal{H}_0 \supset \mathcal{H}_1 \supset \ldots \supset \mathcal{H}_{r-1} \supset \mathcal{H}_r = 0, \qquad \dim \mathcal{H}_{k-1}/\mathcal{H}_k = 1, \qquad k = 1,\ldots,r.$$

Hence, we can construct an adapted basis $\{h_1,\ldots,h_r\}$ of $\mathcal{H}_\Lambda$ such that $\{h_k,\ldots,h_r\}$ form a basis for $\mathcal{H}_{k-1}$ for $k = 1,\ldots,r$. If $\alpha < \beta$, we have $\mathcal{H}_\alpha \supset \mathcal{H}_\beta$ and, as $\mathcal{H}_\beta$ is an ideal, we see that $\{h_\alpha, h_\beta\}_\Lambda \in \mathcal{H}_\beta$. Consequently, if we write $f_t = \sum_{\alpha=1}^r f_\alpha(t)h_\alpha$ and $\{h_\alpha, h_\beta\}_\Lambda = \sum_{\gamma=1}^r c_{\alpha\beta\gamma}h_\gamma$, equation (14) reads

$$\sum_{\alpha=1}^r \left(\frac{\mathrm{d}f_\alpha}{\mathrm{d}t}h_\alpha + \sum_{\beta=1}^r b_\beta f_\alpha \{h_\alpha, h_\beta\}_\Lambda\right) = \sum_{\alpha=1}^r \left(\frac{\mathrm{d}f_\alpha}{\mathrm{d}t} + \sum_{\beta,\gamma=1}^r b_\beta f_\gamma c_{\gamma\beta\alpha}\right) h_\alpha = 0,$$

where $c_{\gamma\beta\alpha} = 0$ for $\alpha < \max(\gamma,\beta)$. Therefore, we obtain the easily integrable system

$$\frac{\mathrm{d}f_\alpha}{\mathrm{d}t} = \sum_{\beta=1}^r T_{\alpha\beta}(t)f_\beta, \quad \alpha = 1,\ldots,r, \quad \Longrightarrow \quad \frac{\mathrm{d}f}{\mathrm{d}t} = T(t)f, \qquad f \in \mathcal{H}_\Lambda,$$

where $T(t)$ is a lower triangular $r \times r$ matrix with entries $T_{\alpha\gamma}(t) = -\sum_{\beta=1}^r b_\beta(t)c_{\gamma\beta\alpha}$ and $f = (f_1,\ldots,f_r)^T$ is considered as an $r \times 1$ matrix.

The previous method can be generalised for *any* Lie–Hamilton algebra. Let us use a *Levi decomposition* [43] for $(\mathcal{H}_\Lambda, \{\cdot,\cdot\}_\Lambda)$ by writing $\mathcal{H}_\Lambda = \left(\mathcal{H}_{s_1} \oplus \ldots \oplus \mathcal{H}_{s_p}\right) \oplus_s \mathcal{H}_\rho$, where $\mathcal{H}_{s_1} \oplus \ldots \oplus \mathcal{H}_{s_p}$ is a direct sum of simple subalgebras of $\mathcal{H}_\Lambda$, the Lie subalgebra $\mathcal{H}_\rho$ is its *radical*, i.e. its maximal solvable ideal, and $\oplus_s$ stands for a semidirect sum. Let us also denote $\lambda_i = \dim \mathcal{H}_{s_i}$ for $i = 1, \ldots, p$ and $r = \dim \mathcal{H}_\rho$. Then, every element $f \in \mathcal{H}_\Lambda$ can be written in a unique way as $f_\rho + f_{s_1} + \ldots + f_{s_p}$, where $f_\rho \in \mathcal{H}_\rho$ and $f_{s_i} \in \mathcal{H}_{s_i}$ for $i = 1, \ldots, p$. Proceeding as before, we can choose a basis $h_1, \ldots, h_r$ adapted to a flag decomposition of the solvable Lie algebra $\mathcal{H}_\rho$ and extend it to a basis of $\mathcal{H}_\Lambda$ by choosing a basis $h_1^{(s_i)}, \ldots, h_{\lambda_i}^{(s_i)}$ for each $\mathcal{H}_{s_i}$. In this way, we obtain that

$$\frac{\mathrm{d}f_\rho}{\mathrm{d}t} = T(t) f_\rho + \sum_{i=1}^{p} B_i(t) f_{s_i}, \qquad \frac{\mathrm{d}f_{s_j}}{\mathrm{d}t} = C_j(t) f_{s_j}, \quad j = 1, \ldots, p, \tag{15}$$

where, as before, $T(t)$ is a lower triangular $r \times r$ matrix, while $B_i(t)$ and $C_i(t)$ are $r \times \lambda_i$ and $\lambda_i \times \lambda_i$ matrices with t-dependent coefficients for $i = 1, \ldots, p$. Consequently, the initial system (14) reduces to working out the equations for $f_{s_1}, \ldots, f_{s_p}$ in each $\mathcal{H}_{s_i}$ separately. From that, f_ρ can be determined through an affine system whose homogeneous associated system is easily integrable.

If $\mathcal{H}_\Lambda$ is a *simple* Lie algebra, (15) is just an expression in coordinates of the system (14), which does not provide any simplification. However, when $\mathcal{H}_\Lambda$ is a *non-simple* one, our techniques do simplify the calculation of Lie integrals. Although similar results can be achieved by using the procedure developed in [42] that, nevertheless, requires the use of certain transformations and integrations of ODEs, which are here unnecessary.

5.3 Polynomial Lie integrals and Casimir functions

We here investigate a generalization of Lie integrals: the hereafter called *polynomial Lie integrals*. Although we prove that these constants of motion can be determined by Lie integrals, we also show that their determination can be simpler in some cases. In particular, we can obtain polynomial Lie integrals algebraically by means of the Casimir functions related to the Lie algebra of Lie integrals.

Definition 16. Let X be a Lie–Hamilton system admitting a compatible Lie–Hamiltonian structure (N, Λ, h). A *polynomial Lie integral* for X with respect to (N, Λ, h) is a constant of motion f for X of the form $f_t = \sum_{I \in M} \lambda_I(t) h^I$, where the I's are r-multi-indexes, i.e. sets $(i_1, \ldots, i_r)$ of nonnegative integers, the set M is a finite family of multi-indexes, the $\lambda_I(t)$ are certain t-dependent functions, and $h^I = h_1^{i_1} \cdot \ldots \cdot h_r^{i_r}$ for a fixed basis $\{h_1, \ldots, h_r\}$ for $\mathcal{H}_\Lambda$.

The study of polynomial Lie integrals can be approached through the symmetric Lie algebra $S_\mathfrak{g}$, where $\mathfrak{g} \simeq \mathcal{H}_\Lambda$.

Lemma 17. *Every Lie algebra isomorphism $\phi : (\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}) \to (\mathcal{H}_\Lambda, \{\cdot,\cdot\}_\Lambda)$ can be extended in a unique way to a Poisson algebra morphism $D : (S_\mathfrak{g}, \cdot, \{\cdot,\cdot\}_{S_\mathfrak{g}}) \to (C^\infty(N), \cdot, \{\cdot,\cdot\}_\Lambda)$. Indeed, if $\{v_1, \ldots, v_r\}$ is a basis for $\mathfrak{g}$, then $D(P(v_1, \ldots, v_r)) = P(\phi(v_1), \ldots, \phi(v_r))$, for every polynomial $P \in S_\mathfrak{g}$.*

The proof is addressed in the Appendix. Recall that "$\cdot$" denotes the standard product of elements of $S_{\mathfrak{g}}$ understood as polynomial functions on $S_{\mathfrak{g}}$. It is remarkable that D does not need to be injective, which causes that $S_{\mathfrak{g}}$ is not in general isomorphic to the space of polynomials on the elements of a basis of $\mathcal{H}_\Lambda$. For instance, consider the Lie algebra isomorphism $\phi : (\mathfrak{sl}(2,\mathbb{R}), [\cdot,\cdot]_{\mathfrak{sl}(2,\mathbb{R})}) \to (\mathcal{H}_\Lambda, \{\cdot,\cdot\})_\Lambda$, with $\{v_1, v_2, v_3\}$ being a basis of $\mathfrak{sl}(2,\mathbb{R})$, of the form $\phi(v_1) = p^2$, $\phi(v_2) = xp$ and $\phi(v_3) = x^2$ and $\{\cdot,\cdot\}$ being the standard Poisson structure on $\mathrm{T}^*\mathbb{R}$. Then, $D(v_1v_3 - v_2^2) = \phi(v_1)\phi(v_3) - \phi^2(v_2) = 0$.

The following notion allows us to simplify the statement and proofs of our results.

Definition 18. Given a curve P_t in $S_{\mathfrak{g}}$, its *degree*, $\deg(P_t)$, is the highest degree of the polynomials $\{P_t\}_{t\in\mathbb{R}}$. If there exists no finite highest degree, we say that $\deg(P_t) = \infty$.

Proposition 19. *A function f is a polynomial Lie integral for a Lie–Hamilton system X with respect to the Lie–Hamiltonian structure (N, Λ, h) if and only if for every $t \in \mathbb{R}$ we have $f_t = D(P_t)$, where D is the Poisson algebra morphism $D : (S_{\mathfrak{g}}, \cdot, \{\cdot,\cdot\}_{S_{\mathfrak{g}}}) \to (C^\infty(N), \cdot, \{\cdot,\cdot\}_\Lambda)$ induced by $\phi : (\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}) \to (\mathcal{H}_\Lambda, \{\cdot,\cdot\}_\Lambda)$, and the curve P_t is a solution of finite degree for*

$$\frac{\mathrm{d}P}{\mathrm{d}t} + \{P, w_t\}_{S_{\mathfrak{g}}} = 0, \qquad P \in S_{\mathfrak{g}}, \tag{16}$$

where w_t stands for a curve in $\mathfrak{g}$ such that $D(w_t) = h_t$ for every $t \in \mathbb{R}$.

Proof. Let P_t be a particular solution of (16). Since D is a Poisson algebra morphism and $h_t = D(w_t)$ for every $t \in \mathbb{R}$, we obtain by applying D to (16) that

$$\frac{\partial D(P_t)}{\partial t} + \{D(P_t), D(w_t)\}_{\bar{\Lambda}} = \frac{\partial D(P_t)}{\partial t} + \{D(P_t), h_t\}_{\bar{\Lambda}} = 0. \tag{17}$$

So, $D(P_t)$ is a Lie integral for X. Note that this does not depend on the chosen curve w_t satisfying $D(w_t) = h_t$.

Conversely, given a polynomial Lie integral f for X, there exists a curve P_t of finite degree such that $D(P_t) = f_t$ for every $t \in \mathbb{R}$. Hence, we see that

$$D\left(\frac{\mathrm{d}P_t}{\mathrm{d}t} + \{P_t, w_t\}_{S_{\mathfrak{g}}}\right) = \frac{\partial D(P_t)}{\partial t} + \{D(P_t), D(w_t)\}_{\bar{\Lambda}} = 0 \Longrightarrow \frac{\mathrm{d}P_t}{\mathrm{d}t} + \{P_t, w_t\}_{S_{\mathfrak{g}}} = \xi_t,$$

where ξ_t is a curve in $\ker D$. As $\deg(dP_t/\mathrm{d}t)$ and $\deg(\{P_t, w_t\}_{S_{\mathfrak{g}}})$ are at most $\deg(P_t)$, then $\deg(\xi_t) \leq \deg(P_t)$. Next, consider the equation

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} + \{\eta, w_t\}_{S_{\mathfrak{g}}} = \xi_t, \qquad \deg(\eta) \leq \deg(P) \quad \text{and} \quad \eta \subset \ker D.$$

Note that this equation is well defined. Indeed, since $\deg(\eta) \leq \deg(P)$ and $\deg(w_t) \leq 1$ for every $t \in \mathbb{R}$, then $\deg(\{\eta, w_t\}_{S_{\mathfrak{g}}}) \leq \deg(P)$ for all $t \in \mathbb{R}$. In addition, as $D(\eta_t) \subset \ker D$, then $\{\eta, w_t\}_{S_{\mathfrak{g}}} \in \ker D$. Then, the above equation can be restricted to the finite-dimensional space of elements of $\ker D$ with degree at most $\deg(P_t)$. Given a particular solution η_t of this equation, which exists for the Theorem of existence and uniqueness, we have that $P_t - \eta_t$ is a solution of (16) projecting into f_t. □

Proposition 20. *Every polynomial Lie integral f of a Lie–Hamilton system X admitting a Lie–Hamiltonian structure (N, Λ, h) can be brought into the form $f = \sum_{I\in M} c_I l^I$, where M is a finite set of multi-indexes, the c_I's are certain real constants, and $l^I = f_1^{i_1} \cdot \ldots \cdot f_r^{i_r}$, with $f_1, \ldots, f_r$ being a basis of the space $\mathfrak{L}_h^\Lambda$.*

Proof. From Proposition 19, we have that $f_t = D(P_t)$ for a solution P_t of finite degree p for (16). So, it is a solution of the restriction of this system to $\mathbb{P}(p, \mathfrak{g})$, i.e. the elements of $S_\mathfrak{g}$ of degree at most p. Given the isomorphism $\phi : \mathfrak{g} \to \mathcal{H}_\Lambda$, define $\phi^{-1}(f_j)$, with $j = 1, \ldots, r$, to be the curve in $\mathfrak{g}$ of the form $t \mapsto \phi^{-1}(f_j)_t$. Note that $v_1 \equiv \phi^{-1}(f_1)_0, \ldots, v_r \equiv \phi^{-1}(f_r)_0$ form a basis of $\mathfrak{g}$. Hence, their polynomials up to order p span a basis for $\mathbb{P}(p, \mathfrak{g})$ and we can write $P_0 = \sum_{I\in M} c_I v^I$, where $v^I = v_1^{i_1} \cdot \ldots \cdot v_r^{i_r}$. As $P_t' = \sum_{I\in M} c_I [\phi^{-1}(f_1)]_t^{i_1} \cdots [\phi^{-1}(f_r)]_t^{i_r}$ and P_t are solutions with the same initial condition of the restriction of (16) to $\mathbb{P}(p, \mathfrak{g})$, they must be the same in virtue of the Theorem of existence and uniqueness of systems of differential equations. Applying D, we obtain that $f_t = D(P_t) = D(\sum_{I\in M} c_I [\phi^{-1}(f_1)_t]^{i_1} \cdots [\phi^{-1}(f_r)_t]^{i_r}) = \sum_{I\in M} c_I l_t^I$. □

Corollary 21. *Let X be a Lie–Hamilton system that possesses a Lie–Hamiltonian structure (N, Λ, h) inducing a Lie algebra isomorphism $\phi : \mathfrak{g} \to \mathcal{H}_\Lambda$ and a Poisson algebra morphism $D : S_\mathfrak{g} \to C^\infty(N)$. The function $F = D(C)$, where C is a Casimir element of $S_\mathfrak{g}$, is a t-independent constant of motion of X. If $\mathcal{C}$ is a Casimir element of $U_\mathfrak{g}$, then $F = D(\lambda^{-1}(\mathcal{C}))$ is t-independent constant of motion for X.*

Note that if C is a constant of motion for X, it is also so for any other X' whose $V^{X'} \subset V^X$. From Proposition 20 and Corollary 21, it follows that $F = D(C) = \sum_{I\in M} c_I l^I$. Therefore, the knowledge of Casimir elements provides not only constants of motion for Lie–Hamilton systems, but also information about the Lie integrals of the system.

As Casimir functions are known for many Lie algebras, we can use them to derive constants of motion for the corresponding Lie–Hamilton systems algebraically instead of applying the usual procedure, i.e. by solving a system of PDEs or ODEs.

In particular, Casimir functions for (semi)simple Lie algebras of arbitrary dimension are known [44, 45]. The same is true for the so-called "quasi-simple" Lie algebras, which can be obtained from simple Lie algebras through contraction techniques [46]. Moreover, the Casimir invariants (Casimir elements of the Poisson algebra $(C^\infty(\mathfrak{g}^*), \{\cdot,\cdot\})$) for all real Lie algebras with dimension $d \leq 5$ were given in [47] (recall that the Casimir invariants for some of the solvable cases are not polynomial, i.e. they cannot be considered as elements of $S_\mathfrak{g}$), and the literature dealing with Casimir invariants for solvable and nilpotent Lie algebras is not scarce (see, e.g. [48, 49, 50]).

6 Superposition rules from Poisson coalgebras

We here prove that each Lie–Hamiltonian structure (N, Λ, h) for a Lie–Hamilton system X gives rise in a natural way to a Poisson coalgebra $(S_\mathfrak{g}, \cdot, \{\cdot,\cdot\}_{S_\mathfrak{g}}, \Delta)$ where $\mathfrak{g} \simeq (\mathcal{H}_\Lambda, \{\cdot,\cdot\}_\Lambda)$. This allows us to use the coproduct of this coalgebra to construct new Lie–Hamiltonian structures for all the diagonal prolongations of X and to derive algebraically t-independent constants of motion for such diagonal prolongations. In turn, these constants can further be employed to obtain a superposition rule for the initial system. Our findings, which are only applicable to "primitive" Poisson coalgebras, are rigorous demonstrations and generalisations of previous achievements established in [19, 26, 51].

6.1 Poisson coalgebras for Lie–Hamiltonian structures

Lemma 22. *Given a Lie–Hamilton system X with a Lie–Hamiltonian structure (N,Λ,h), the space $(S_\mathfrak{g},\cdot,\{\cdot,\cdot\}_{S_\mathfrak{g}},\Delta)$, with $\mathfrak{g}\simeq(\mathcal{H}_\Lambda,\{\cdot,\cdot\}_\Lambda)$, is a Poisson coalgebra with a coproduct $\Delta:S_\mathfrak{g}\to S_\mathfrak{g}\otimes S_\mathfrak{g}$ satisfying*

$$\Delta(v)=v\otimes 1+1\otimes v,\qquad \forall v\in\mathfrak{g}\subset S_\mathfrak{g}. \tag{18}$$

Proof. We know that $(S_\mathfrak{g},\cdot,\{\cdot,\cdot\}_{S_\mathfrak{g}})$ and $(S_\mathfrak{g}\otimes S_\mathfrak{g},\cdot_{S_\mathfrak{g}\otimes S_\mathfrak{g}},\{\cdot,\cdot\}_{S_\mathfrak{g}\otimes S_\mathfrak{g}})$ are Poisson algebras. The coassociativity property for the coproduct map (18) is straightforward. Therefore, let us prove that there exists a Poisson algebra morphism $\Delta:(S_\mathfrak{g},\cdot,\{\cdot,\cdot\}_{S_\mathfrak{g}})\to(S_\mathfrak{g}\otimes S_\mathfrak{g},\cdot_{S_\mathfrak{g}\otimes S_\mathfrak{g}},\{\cdot,\cdot\}_{S_\mathfrak{g}\otimes S_\mathfrak{g}})$ satisfying (18), which turns $(S_\mathfrak{g},\cdot,\{\cdot,\cdot\}_{S_\mathfrak{g}},\Delta)$ into a Poisson coalgebra.

The elements of $S_\mathfrak{g}$ of the form $v^I\equiv v_1^{i_1}\cdot\ldots\cdot v_r^{i_r}$, where the I's are r-multi-index with $r=\dim\mathfrak{g}$, form a basis for $S_\mathfrak{g}$ (considered as a linear space). Then, every $P\in S_\mathfrak{g}$ can be written in a unique way as $P=\sum_{I\in M}\lambda_I v^I$, where the λ_I are real constants and I runs all the elements of a finite set M. In view of this, an $\mathbb{R}$-algebra morphism $\Delta:S_\mathfrak{g}\to S_\mathfrak{g}\otimes S_\mathfrak{g}$ is determined by the image of the elements of a basis for $\mathfrak{g}$, i.e.

$$\Delta(P)=\sum_I\lambda_I\Delta(v^I)=\sum_I\lambda_I\Delta(v_1^{i_1})\cdot\ldots\cdot\Delta(v_r^{i_r}). \tag{19}$$

Therefore, two $\mathbb{R}$-algebra morphisms that coincide on the elements on $\mathfrak{g}$ are necessarily the same. Hence, if there exists such a morphism, it is unique. Let us prove that there exists an $\mathbb{R}$-algebra morphism Δ satisfying (18).

From (19), we easily see that Δ is $\mathbb{R}$-linear. Let us also prove that $\Delta(PQ)=\Delta(P)\Delta(Q)$ for every $P,Q\in S_\mathfrak{g}$, which shows that Δ is an $\mathbb{R}$-algebra morphism. If we write $Q=\sum_{J\in M}\lambda_J v^J$, we obtain that

$$\Delta(PQ)=\sum_K\left(\sum_{I+J=K}\lambda_I\lambda_J\right)\Delta(v^K)=\sum_I\lambda_I\Delta(v^I)\sum_J\lambda_J\Delta(v^J)=\Delta(P)\Delta(Q).$$

Finally, we show that Δ is also a Poisson morphism. By linearity, this reduces to proving that $\Delta(\{v^I,v^J\}_{S_\mathfrak{g}})=\{\Delta(v^I),\Delta(v^J)\}_{S_\mathfrak{g}\otimes S_\mathfrak{g}}$. If $|I|=0$ or $|J|=0$, this result is immediate as the Poisson bracket involving a constant is zero. For the remaining cases and starting by $|I|+|J|=2$, we have that $\Delta(\{v_\alpha,v_\beta\}_{S_\mathfrak{g}})=\{\Delta(v_\alpha),\Delta(v_\beta)\}_{S_\mathfrak{g}\otimes S_\mathfrak{g}}$, $\forall\alpha,\beta=1,\ldots,r$. Proceeding by induction, we prove that this holds for every value of $|I|+|J|$; by writing $v^I=v^{\bar I}v_\gamma^{i_\gamma}$ and using induction hypothesis, we get

$$\Delta\left(\{v^I,v^J\}_{S_\mathfrak{g}}\right)\quad=\Delta\left(\{v^{\bar I}v_\gamma^{i_\gamma},v^J\}_{S_\mathfrak{g}}\right)=\Delta\left(\{v^{\bar I},v^J\}_{S_\mathfrak{g}}v_\gamma^{i_\gamma}+v^{\bar I}\{v_\gamma^{i_\gamma},v^J\}_{S_\mathfrak{g}}\right)$$

$$= \left\{\Delta(v^{\bar{I}}), \Delta(v^J)\right\}_{S_{\mathfrak{g}}\otimes S_{\mathfrak{g}}}\Delta(v_\gamma^{i_\gamma}) + \Delta(v^{\bar{I}})\left\{\Delta(v_\gamma^{i_\gamma}), \Delta(v^J)\right\}_{S_{\mathfrak{g}}\otimes S_{\mathfrak{g}}}$$
$$= \left\{\Delta(v^{\bar{I}})\Delta(v_\gamma^{i_\gamma}), \Delta(v^J)\right\}_{S_{\mathfrak{g}}\otimes S_{\mathfrak{g}}} = \left\{\Delta(v^I), \Delta(v^J)\right\}_{S_{\mathfrak{g}}\otimes S_{\mathfrak{g}}}.$$

□

The coproduct defined in the previous lemma gives rise to a new Poisson algebra morphism as stated in the following immediate lemma.

Lemma 23. *The map* $\Delta^{(m)} : (S_{\mathfrak{g}}, \cdot, \{\cdot,\cdot\}_{S_{\mathfrak{g}}}) \to (S_{\mathfrak{g}}^{(m)}, \cdot_{S_{\mathfrak{g}}^{(m)}}, \{\cdot,\cdot\}_{S_{\mathfrak{g}}^{(m)}})$ *, with* $m > 1$*, defined by recursion following (7) with* $\Delta^{(2)} = \Delta$ *given by (18), is a Poisson algebra morphism.*

The injection $\iota : \mathfrak{g} \to \mathcal{H}_\Lambda \subset C^\infty(N)$ is a Lie algebra morphism that can be extended to a Poisson algebra morphism $D : S_{\mathfrak{g}} \to C^\infty(N)$ of the form $D(P(v_1,\ldots,v_r)) = P(\iota(v_1),\ldots,\iota(v_r))$. Recall that this map need not to be injective.

Lemma 24. *The Lie algebra morphism* $\mathfrak{g} \hookrightarrow C^\infty(N)$ *gives rise to a family of Poisson algebra morphisms* $D^{(m)} : S_{\mathfrak{g}}^{(m)} \hookrightarrow C^\infty(N)^{(m)} \subset C^\infty(\,N^m\,)$ *satisfying, for all* $v_1,\ldots,v_m \in \mathfrak{g} \subset S_{\mathfrak{g}}$*, that*

$$\left[D^{(m)}(v_1\otimes\ldots\otimes v_m)\right](x_{(1)},\ldots,x_{(m)}) = [D(v_1)](x_{(1)})\cdot\ldots\cdot[D(v_m)](x_{(m)}), \tag{20}$$

where $x_{(i)}$ *is a point of the manifold* N *placed in the* i*-position within the product* $N\times\ldots\times N \equiv N^m$*.*

6.2 Constants of motion from Poisson coalgebras

From the above results, we can easily demonstrate the following statement which shows that the diagonal prolongations of a Lie–Hamilton system X are also Lie–Hamilton ones admitting a structure induced by that of X.

Proposition 25. *If* X *is a Lie–Hamilton system on* N *with a Lie–Hamiltonian structure* (N,Λ,h)*, then the diagonal prolongation* $\widetilde{X}$ *to each* N^{m+1} *is also a Lie–Hamilton system endowed with a Lie–Hamiltonian structure* $(N^{m+1},\Lambda^{m+1},\widetilde{h})$ *given by* $\Lambda^{m+1}(x_{(0)},\ldots,x_{(m)}) = \sum_{a=0}^m \Lambda(x_{(a)})$*, where we make use of the vector bundle isomorphism* $\mathrm{T}N^{m+1} \simeq \mathrm{T}N\oplus\cdots\oplus\mathrm{T}N$*, and* $\widetilde{h}_t = D^{(m+1)}(\Delta^{(m+1)}(h_t))$*, where* $D^{(m+1)}$ *is the Poisson algebra morphism (20) induced by the Lie algebra morphism* $\mathfrak{g}\hookrightarrow\mathcal{H}_\Lambda\subset C^\infty(N)$*.*

The above results enable us to prove the following theorem that provides a method to obtain t-independent constants of motion for the diagonal prolongations of a Lie–Hamilton system. From this theorem, one may obtain superposition rules for Lie–Hamilton systems in an algebraic way. Additionally, this theorem is a generalization, only valid in the case of primitive coproduct maps, of the integrability theorem for coalgebra symmetric systems given in [26].

Theorem 26. *If X is a Lie–Hamilton system with a Lie–Hamiltonian structure (N, Λ, h) and C is a Casimir element of $(S_{\mathfrak{g}}, \cdot, \{,\}_{S_{\mathfrak{g}}})$, where $\mathfrak{g} \simeq \mathcal{H}_\Lambda$, then:*
(i) The functions defined as

$$F^{(k)} = D^{(k)}(\Delta^{(k)}(C)), \qquad k = 2, \ldots, m, \tag{21}$$

are t-independent constants of motion for the diagonal prolongation $\widetilde{X}$ to N^m. Furthermore, if all the $F^{(k)}$ are non-constant functions, they form a set of $(m-1)$ functionally independent functions in involution.
(ii) The functions given by

$$F_{ij}^{(k)} = S_{ij}(F^{(k)}), \qquad 1 \leq i < j \leq k, \qquad k = 2, \ldots, m, \tag{22}$$

where S_{ij} is the permutation of variables $x_{(i)} \leftrightarrow x_{(j)}$, are t-independent constants of motion for the diagonal prolongation $\widetilde{X}$ to N^m.

Proof. Every $P \in S_{\mathfrak{g}}^{(j)}$ can naturally be considered as an element $P \otimes \overbrace{1 \otimes \ldots \otimes 1}^{(k-j)-\text{times}} \in S_{\mathfrak{g}}^{(k)}$. Since $j \leq k$, we have that $\{\Delta^{(j)}(\bar{v}), \Delta^{(k)}(v)\}_{S_{\mathfrak{g}}^{(k)}} = \{\Delta^{(j)}(\bar{v}), \Delta^{(j)}(v)\}_{S_{\mathfrak{g}}^{(j)}}$, $\forall \bar{v}, v \in \mathfrak{g}$. So,

$$\{\Delta^{(j)}(C), \Delta^{(k)}(v)\}_{S_{\mathfrak{g}}^{(k)}} = \{\Delta^{(j)}(C), \Delta^{(j)}(v)\}_{S_{\mathfrak{g}}^{(j)}} = \Delta^{(j)}\left(\{C, v\}_{S_{\mathfrak{g}}^{(j)}}\right) = 0.$$

Hence, by using that every function $f \in C^\infty(N^j)$ can be understood as a function $\pi^* f \in C^\infty(N^k)$, being $\pi : N^j \times N^{k-j} \to N^j$ the projection onto the first factor, and by applying the Poisson algebra morphisms introduced in Lemma 24 we get

$$\left\{D^{(j)}(\Delta^{(j)}(C)), D^{(k)}(\Delta^{(k)}(v))\right\}_{\Lambda^k} = \left\{F^{(j)}, D^{(k)}(\Delta^{(k)}(v))\right\}_{\Lambda^k} = 0, \;\; \forall v \in \mathfrak{g},$$

which leads to $\left\{F^{(j)}, F^{(k)}\right\}_{\Lambda^k} = 0$, that is, the functions (21) are in involution. By construction (see Lemma 23), if these are non-constant, then they are functionally independent functions since $F^{(j)}$ lives in N^j, meanwhile $F^{(k)}$ is defined on N^k.

Let us prove now that all the functions (21) and (22) are t-independent constants of motion for $\widetilde{X}$. Using that $\mathcal{H}_\Lambda \simeq \mathfrak{g}$ and $X_t = -\widehat{\Lambda} \circ dh_t$, we see that X can be brought into the form $X_t = -\widehat{\Lambda} \circ d \circ D(v_t)$ for a unique curve $t \to v_t$ in $\mathfrak{g}$. From this and Proposition 25, it follows

$$\widetilde{X}_t = -\Lambda^m \circ dD^{(m)}(\Delta^{(m)}(v_t)) \Longrightarrow \widetilde{X}_t(F^{(k)}) = \left\{D^{(k)}(\Delta^{(k)}(C)), D^{(m)}(\Delta^{(m)}(v_t))\right\}_{\Lambda^m} = 0.$$

Then, $F^{(k)}$ is a common first-integral for every $\widetilde{X}_t$. Finally, consider the permutation operators S_{ij}, with $1 \leq i < j \leq k$ for $k = 2, \ldots, m$. Note that

$$\begin{aligned} 0 = S_{ij}\left\{F^{(k)}, D^{(m)}(\Delta^{(m)}(v_t))\right\}_{\Lambda^m} &= \left\{S_{ij}(F^{(k)}), S_{ij}\left(D^{(m)}(\Delta^{(m)}(v_t))\right)\right\}_{\Lambda^m} \\ &= \left\{F_{ij}^{(k)}, D^{(m)}(\Delta^{(m)}(v_t))\right\}_{\Lambda^m} = \widetilde{X}_t(F_{ij}^{(k)}). \end{aligned}$$

Consequently, the functions $F_{ij}^{(k)}$ are t-independent constants of motion for $\widetilde{X}$. $\square$

Note that the "omitted" case with $k=1$ in the set of constants (21) is, precisely, the one provided by Corollary 21 as $F^{(1)} \equiv F = D(C)$. Depending on the system X, or more specifically, on the associated $\mathcal{H}_\Lambda$, the function F can be either a useless trivial constant or a relevant function. It is also worth noting that constants (22) need not be functionally independent, but we can always choose those fulfilling such a property. Finally, observe that if X' is such that $V^{X'} \subset V^X$, then the functions (21) and (22) are also constants of motion for the diagonal prolongation of X' to N^m.

7 Applications

Let us illustrate our methods through the analysis of systems of physical and mathematical relevance. We firstly use the techniques described in Section 5.3 to easily retrive the so-called Lewis–Riesenfeld invariant for Ermakov systems [52] and the t-independent constant of motion for a system of Riccati equations employed to construct the superposition rule for Riccati equations [7]. Secondly, we use the coalgebra approach developed in Section 6 to deduce superposition rules for Kummer–Schwarz equations [13, 53], Smorodinsky–Winternitz systems [22, 23] with time-dependent frequency and also for a classical system with trigonometric nonlinearities, whose integrability has recently drawn some attention [18, 54]. As we avoid the integration of PDEs or ODEs used in standard methods, our techniques are simpler than previous ones.

7.1 Classical Ermakov systems

Let us consider the classical Ermakov system [9]:

$$\begin{cases} \frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -\omega^2(t)x + \frac{b}{x^3}, \\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -\omega^2(t)y, \end{cases}$$

with a non-constant t-dependent frequency $\omega(t)$, being b a real constant. This system appears in a number of applications related to problems in quantum and classical mechanics [52]. By writting this system as a first-order one

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} &= v_x, \qquad & \frac{\mathrm{d}v_x}{\mathrm{d}t} &= -\omega^2(t)x + \frac{b}{x^3}, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= v_y, \qquad & \frac{\mathrm{d}v_y}{\mathrm{d}t} &= -\omega^2(t)y, \end{cases} \tag{23}$$

we can apply the theory of Lie systems. Indeed, this is a Lie system related to a Vessiot–Guldberg Lie algebra V isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ [10]. In fact, system (23) describes the integral curves of the t-dependent vector field $X = X_3 + \omega^2(t)X_1$, where the vector fields

$$X_1 = -x\frac{\partial}{\partial v_x} - y\frac{\partial}{\partial v_y}, \; X_2 = \frac{1}{2}\left(v_x\frac{\partial}{\partial v_x} + v_y\frac{\partial}{\partial v_y} - x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right), X_3 = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + \frac{b}{x^3}\frac{\partial}{\partial v_x},$$

satisfy the commutation relations

$$[X_1, X_2] = X_1, \qquad [X_1, X_3] = 2X_2, \qquad [X_2, X_3] = X_3. \tag{24}$$

As a first new result we show that this is a Lie–Hamilton system. The vector fields are Hamiltonian with respect to the Poisson bivector $\Lambda = \partial/\partial x \wedge \partial/\partial v_x + \partial/\partial y \wedge \partial/\partial v_y$ provided that $X_\alpha = -\widehat{\Lambda}(dh_\alpha)$ for $\alpha = 1,2,3$. Thus, we find the following Hamiltonian functions which form a basis for $(\mathcal{H}_\Lambda, \{\cdot,\cdot\}_\Lambda) \simeq (\mathfrak{sl}(2,\mathbb{R}), [\cdot,\cdot])$:

$$h_1 = \frac{1}{2}(x^2+y^2), \qquad h_2 = -\frac{1}{2}(xv_x + yv_y), \qquad h_3 = \frac{1}{2}\left(v_x^2 + v_y^2 + \frac{b}{x^2}\right),$$

as they fulfil

$$\{h_1,h_2\} = -h_1, \qquad \{h_1,h_3\} = -2h_2, \qquad \{h_2,h_3\} = -h_3. \tag{25}$$

Since $X = X_3 + \omega^2(t)X_1$ and $\omega(t)$ is not a constant, every t-independent constant of motion f for X is a common first-integral for X_1, X_2, X_3. Instead of searching an f by solving the system of PDEs given by $X_1 f = X_2 f = X_3 f = 0$, we use Corollary 21. This easily provides such a first integral through the Casimir element of the symmetric algebra of $\mathfrak{sl}(2,\mathbb{R})$. Explicitly, given a basis $\{v_1,v_2,v_3\}$ for $\mathfrak{sl}(2,\mathbb{R})$ satisfying

$$[v_1,v_2] = -v_1, \qquad [v_1,v_3] = -2v_2, \qquad [v_2,v_3] = -v_3, \tag{26}$$

the Casimir element of $\mathfrak{sl}(2,\mathbb{R})$ reads $\mathcal{C} = \frac{1}{2}(v_1\widetilde{\otimes}v_3 + v_3\widetilde{\otimes}v_1) - v_2\widetilde{\otimes}v_2 \in U_{\mathfrak{sl}(2,\mathbb{R})}$. Then, the inverse of symmetrizer morphism (5), $\lambda^{-1} : U_{\mathfrak{sl}(2,\mathbb{R})} \to S_{\mathfrak{sl}(2,\mathbb{R})}$, gives rise to the Casimir element of $S_{\mathfrak{sl}(2,\mathbb{R})}$:

$$C = \lambda^{-1}(\mathcal{C}) = v_1v_3 - v_2^2. \tag{27}$$

According to Lemma 17 we consider the Poisson algebra morphism D induced by the isomorphism $\phi : \mathfrak{sl}(2,\mathbb{R}) \to \mathcal{H}_\Lambda$ defined by $\phi(v_\alpha) = h_\alpha$ for $\alpha = 1,2,3$. Subsequently, via Corollary 21, we obtain

$$F = D(C) = \phi(v_1)\phi(v_3) - \phi^2(v_2) = h_1h_3 - h_2^2 = (v_y x - v_x y)^2 + b\left(1 + \frac{y^2}{x^2}\right).$$

In this way, we recover, up to an additive and multiplicative constant, the well-known Lewis–Riesenfeld invariant [52]. Note that when $\omega(t)$ is a constant, then $V^X \subset V$ and the function F is also a constant of motion for X (23).

7.2 Riccati equations

Let us turn to the system of Riccati equations on $\mathcal{O} = \{(x_1,x_2,x_3,x_4) \,|\, x_i \neq x_j, i \neq j = 1,\ldots,4\} \subset \mathbb{R}^4$, given by

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = a_0(t) + a_1(t)x_i + a_2(t)x_i^2, \qquad i = 1,\ldots,4, \tag{28}$$

where $a_0(t), a_1(t), a_2(t)$ are arbitrary t-dependent functions. The knowledge of a non-constant t-independent constant of motion for any system of this type leads to obtaining a superposition rule for Riccati equations [7]. Usually, this requires the integration of a system of PDEs [7] or ODEs [6]. As in the previous subsection, we obtain such a t-independent constant of motion through algebraic methods by showing that (28) is a Lie–Hamilton system with a given Lie–Hamiltonian structure and obtaining an associated polynomial Lie integral.

Observe that (28) is a Lie system related to a t-dependent vector field $X = a_0(t)X_1 + a_1(t)X_2 + a_2(t)X_3$, where

$$X_1 = \sum_{i=1}^{4} \frac{\partial}{\partial x_i}, \qquad X_2 = \sum_{i=1}^{4} x_i \frac{\partial}{\partial x_i}, \qquad X_3 = \sum_{i=1}^{4} x_i^2 \frac{\partial}{\partial x_i}$$

span a Vessiot–Guldberg Lie algebra V for (28) isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ satisfying the same commutation relations (24). For simplicity, we assume $V^X = V$. Nevertheless, our final results are valid for any other case.

To show that (28) is a Lie–Hamilton system for arbitrary functions $a_0(t)$, $a_1(t)$, $a_2(t)$, we need to search for a symplectic form ω such that V consists of Hamiltonian vector fields. By impossing $\mathcal{L}_{X_\alpha}\omega = 0$, for $\alpha = 1, 2, 3$, we obtain the 2-form

$$\omega = \frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2}{(x_1 - x_2)^2} + \frac{\mathrm{d}x_3 \wedge \mathrm{d}x_4}{(x_3 - x_4)^2},$$

which is closed and non-degenerate on $\mathcal{O}$. Now, observe that $\iota_{X_\alpha}\omega = \mathrm{d}h_\alpha$, with $\alpha = 1, 2, 3$ and

$$h_1 = \frac{1}{x_1 - x_2} + \frac{1}{x_3 - x_4}, \quad h_2 = \frac{1}{2}\left(\frac{x_1 + x_2}{x_1 - x_2} + \frac{x_3 + x_4}{x_3 - x_4}\right), \quad h_3 = \frac{x_1 x_2}{x_1 - x_2} + \frac{x_3 x_4}{x_3 - x_4}.$$

So, h_1, h_2 and h_3 are Hamiltonian functions for X_1, X_2 and X_3, correspondingly. Using the Poisson bracket $\{\cdot,\cdot\}_\omega$ induced by ω, we obtain that h_1, h_2 and h_3 satisfy the commutation relations (25), and $(\langle h_1, h_2, h_3\rangle, \{\cdot,\cdot\}_\omega) \simeq \mathfrak{sl}(2,\mathbb{R})$. Next, we again express $\mathfrak{sl}(2,\mathbb{R})$ in the basis $\{v_1, v_2, v_3\}$ with Lie brackets (26) and Casimir function (27), and we consider the Poisson algebra morphism $D : S_{\mathfrak{sl}(2,\mathbb{R})} \to C^\infty(\mathcal{O})$ given by the isomorphism $\phi(v_\alpha) = h_\alpha$ for $\alpha = 1, 2, 3$. As $(\mathcal{O}, \{\cdot,\cdot\}_\omega, h_t = a_0(t)h_1 + a_1(t)h_2 + a_2(t)h_3)$ is a Lie–Hamiltonian structure for X and applying Corollary 21, we obtain the t-independent constant of motion for X:

$$F = D(C) = h_1 h_3 - h_2^2 = \frac{(x_1 - x_4)(x_2 - x_3)}{(x_1 - x_2)(x_3 - x_4)}.$$

As in the previous example, if $V^X \subset V$, then F is also a constant of motion for X. It is worth noting that F is the known constant of motion obtained for deriving a superposition rule for Riccati equations [6, 7], which is here deduced through a simple algebraic calculation.

It is also interesting that V also becomes a Lie algebra of Hamiltonian vector fields with respect to a second symplectic structure given by $\omega = \sum_{1\leq i<j}^{4} \frac{\mathrm{d}x_i \wedge \mathrm{d}x_j}{(x_i - x_j)^2}$. Consequently, the system (28) can be considered, in fact, as a *bi–Lie–Hamilton system.*

7.3 Second-order Kummer–Schwarz equations in Hamiltonian form

It was proved in [16] that the second-order Kummer–Schwarz equations [55, 56] admit a t-dependent Hamiltonian which can be used to work out their Hamilton's equations, namely

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{px^3}{2}, \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= -\frac{3p^2x^2}{4} - \frac{b_0}{4} + \frac{4b_1(t)}{x^2}, \end{cases} \tag{29}$$

where $b_1(t)$ is a non-constant t-dependent function, $(x,p) \in \mathrm{T}^*\mathbb{R}_0$ with $\mathbb{R}_0 \equiv \mathbb{R} - \{0\}$, and b_0 is a real constant. This is a Lie system associated to the t-dependent vector field $X = X_3 + b_1(t)X_1$ [16], where the vector fields

$$X_1 = \frac{4}{x^2}\frac{\partial}{\partial p}, \qquad X_2 = x\frac{\partial}{\partial x} - p\frac{\partial}{\partial p}, \qquad X_3 = \frac{px^3}{2}\frac{\partial}{\partial x} - \frac{1}{4}\left(3p^2x^2 + b_0\right)\frac{\partial}{\partial p}$$

span a Vessiot–Guldberg Lie algebra V isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ fulfilling (24). Moreover, X is a Lie–Hamilton system, as V consists of Hamiltonian vector fields with respect to the Poisson bivector $\Lambda = \partial/\partial x \wedge \partial/\partial p$ on $\mathrm{T}^*\mathbb{R}_0$. Indeed, $X_\alpha = -\widehat{\Lambda}(dh_\alpha)$, with $\alpha = 1,2,3$ and

$$h_1 = \frac{4}{x}, \qquad h_2 = xp, \qquad h_3 = \frac{1}{4}\left(p^2x^3 + b_0x\right) \tag{30}$$

are a basis of a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ satisfying the commutation relations (25). Therefore, (29) is a Lie–Hamilton system possessing a Lie–Hamiltonian structure $(\mathrm{T}^*\mathbb{R}_0, \Lambda, h)$, where $h_t = h_3 + b_1(t)h_1$.

To obtain a superposition rule for X we need to determine an integer m so that the diagonal prolongations of X_α to $\mathrm{T}^*\mathbb{R}_0^m$ $(\alpha = 1,2,3)$ become linearly independent at a generic point (see [7, 13]). This happens for $m = 2$. We consider a coordinate system in $\mathrm{T}\mathbb{R}_0^3$, namely $\{x_{(1)}, p_{(1)}, x_{(2)}, p_{(2)}, x_{(3)}, p_{(3)}\}$. A superposition rule for X can be obtained by determining two common first integrals for the diagonal prolongations $\widetilde{X}_\alpha$ to $\mathrm{T}^*\mathbb{R}_0^3$ satisfying

$$\frac{\partial(F_1, F_2)}{\partial(x_{(1)}, p_{(1)})} \neq 0. \tag{31}$$

Instead of searching F_1, F_2 in the standard way, i.e. by solving the system of PDEs given by $\widetilde{X}_\alpha f = 0$, we make use of Theorem 26. This provides such first integrals through the Casimir element C (27) of the symmetric algebra of $\mathcal{H}_\Lambda \simeq \mathfrak{sl}(2,\mathbb{R})$. Indeed, the coproduct (18) enables us to define the elements

$$\begin{aligned}\Delta(C) &= \Delta(v_1)\Delta(v_3) - \Delta(v_2)^2 = (v_1 \otimes 1 + 1 \otimes v_1)(v_3 \otimes 1 + 1 \otimes v_3) - (v_2 \otimes 1 + 1 \otimes v_2)^2, \\ \Delta^{(3)}(C) &= \Delta^{(3)}(v_1)\Delta^{(3)}(v_3) - \Delta^{(3)}(v_2)^2 = (v_1 \otimes 1 \otimes 1 + 1 \otimes v_1 \otimes 1 + 1 \otimes 1 \otimes v_1) \times \\ &\quad (v_3 \otimes 1 \otimes 1 + 1 \otimes v_3 \otimes 1 + 1 \otimes 1 \otimes v_3) - (v_2 \otimes 1 \otimes 1 + 1 \otimes v_2 \otimes 1 + 1 \otimes 1 \otimes v_2)^2,\end{aligned}$$

for $S^{(2)}_{\mathfrak{sl}(2,\mathbb{R})}$ and $S^{(3)}_{\mathfrak{sl}(2,\mathbb{R})}$, respectively. By applying D, $D^{(2)}$ and $D^{(3)}$ coming from the isomorphism $\phi(v_\alpha) = h_\alpha$ for the Hamiltonian functions (30), we obtain, via Theorem 26, the following constants of motion of the type (21):

$$
\begin{aligned}
&F = D(C) = h_1(x_1,p_1)h_3(x_1,p_1) - h_2^2(x_1,p_1) = b_0,\\
&F^{(2)} = D^{(2)}(\Delta(C)) = (h_1(x_1,p_1)+h_1(x_2,p_2))\,(h_3(x_1,p_1)+h_3(x_2,p_2))\\
&-(h_2(x_1,p_1)+h_2(x_2,p_2))^2 = \tfrac{b_0(x_1+x_2)^2+(p_1x_1^2-p_2x_2^2)^2}{x_1x_2} = \tfrac{b_0(x_1^2+x_2^2)+(p_1x_1^2-p_2x_2^2)^2}{x_1x_2} + 2b_0,\\
&F^{(3)} = D^{(3)}(\Delta(C)) = \textstyle\sum_{i=1}^3 h_1(x_i,p_i)\sum_{j=1}^3 h_3(x_j,p_j) - \left(\sum_{i=1}^3 h_2(x_i,p_i)\right)^2\\
&= \textstyle\sum_{1\le i<j}^3 \tfrac{b_0(x_i+x_j)^2+(p_ix_i^2-p_jx_j^2)^2}{x_ix_j} - 3b_0,
\end{aligned}
$$

where, for the sake of simplicity, hereafter we denote (x_i,p_i) the coordinates $(x_{(i)},p_{(i)})$. Thus F simply gives rise to the constant b_0, while $F^{(2)}$ and $F^{(3)}$ are, by construction, two functionally independent constants of motion for $\widetilde{X}$ fulfilling (31) which, in turn, allows us to to derive a superposition rule for X. Furthermore, the function $F^{(2)} \equiv F^{(2)}_{12}$ provides two other constants of the type (22) given by $F^{(2)}_{13} = S_{13}(F^{(2)})$ and $F^{(2)}_{23} = S_{23}(F^{(2)})$ that verify $F^{(3)} = F^{(2)} + F^{(2)}_{13} + F^{(2)}_{23} - 3b_0$. Since it is simpler to work with $F^{(2)}_{23}$ than with $F^{(3)}$, we choose the pair $F^{(2)}$, $F^{(2)}_{23}$ as the two functionally independent first integrals to obtain a superposition rule. We set

$$
F^{(2)} = k_1 + 2b_0, \qquad F^{(2)}_{23} = k_2 + 2b_0, \tag{32}
$$

and compute x_1, p_1 in terms of the other variables and k_1, k_2. From (32), we have

$$
p_1 = p_1(x_1,x_2,p_2,x_3,p_3,k_1) = \frac{p_2x_2^2 \pm \sqrt{k_1x_1x_2 - b_0(x_1^2+x_2^2)}}{x_1^2}. \tag{33}
$$

Substituying in the second relation within (32), we obtain

$$
x_1 = x_1(x_2,p_2,x_3,p_3,k_1,k_2) = \frac{A^2B_+ + b_0B_-(x_2^2-x_3^2) \pm 2A\sqrt{\Upsilon}}{B_-^2 + 4b_0A^2}, \tag{34}
$$

provided that the functions $A, B_\pm, \Upsilon$ are defined by

$$
\begin{aligned}
&A = p_2x_2^2 - p_3x_3^2, \qquad B_\pm = k_1x_2 \pm k_2x_3,\\
&\Upsilon = A^2\left[k_1k_2x_2x_3 - 2b_0^2(x_2^2+x_3^2) - b_0A^2\right] + b_0x_2x_3B_-(k_2x_2 - k_1x_3) - b_0^3(x_2^2-x_3^2)^2.
\end{aligned}
$$

By introducing this result into (33), we obtain $p_1 = p_1(x_2,p_2,x_3,p_3,k_1,k_2)$ which, along with $x_1 = x_1(x_2,p_2,x_3,p_3,k_1,k_2)$, provides a superposition rule for X.

In particular for (29) with $b_0 = 0$ it results

$$
x_1 = \frac{A^2\left(B_+ \pm 2\sqrt{k_1k_2x_2x_3}\,\right)}{B_-^2}, \quad p_1 = B_-^3\frac{\left(B_-p_2x_2^2 \pm A\sqrt{k_1x_2\left(B_+ \pm 2\sqrt{k_1k_2x_2x_3}\,\right)}\right)}{A^4\left(B_+ \pm 2\sqrt{k_1k_2x_2x_3}\,\right)^2},
$$

where the functions $A, B_\pm$ remain in the above same form. As the constants of motion were derived for non-constant $b_1(t)$, when $b_1(t)$ is constant we have $V^X \subset V$. As a consequence, the functions F, $F^{(2)}$, $F^{(3)}$ and so on are still constants of motion for the diagonal prolongation $\widetilde{X}$ and the superposition rules are still valid for any system (29).

7.4 Smorodinsky–Winternitz systems with a time-dependent frequency

Let us focus on the n-dimensional Smorodinsky–Winternitz systems [22, 23] with unit mass and a non-constant time-dependent frequency $\omega(t)$ whose Hamiltonian is given by

$$h = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\omega^2(t)\sum_{i=1}^{n} x_i^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{b_i}{x_i^2},$$

where the b_i's are n real constants. The corresponding Hamilton's equations read

$$\begin{cases} \frac{\mathrm{d}x_i}{\mathrm{d}t} &= p_i, \\ \frac{\mathrm{d}p_i}{\mathrm{d}t} &= -\omega^2(t)x_i + \frac{b_i}{x_i^3}, \end{cases} \qquad i = 1, \ldots, n. \tag{35}$$

These systems have been recently attracting quite much attention in classical and quantum mechanics for their special properties [16, 57, 58, 59]. Observe that Ermakov systems (23) arise as the particular case of (35) for $n = 2$ and $b_2 = 0$. For $n = 1$ the above system maps into the Milne–Pinney equations, which are of interest in the study of several cosmological models [9, 24, 60], through the diffeomorphism $(x, p) \in \mathrm{T}^*\mathbb{R}_0 \to (x, v = p) \in \mathrm{T}\mathbb{R}_0$.

Let us show that the system (35) can be endowed with a Lie–Hamiltonian structure. This system describes the integral curves of the t-dependent vector field on $\mathrm{T}^*\mathbb{R}_0^n$ given by $X = X_3 + \omega^2(t)X_1$, where the vector fields

$$X_1 = -\sum_{i=1}^{n} x_i\frac{\partial}{\partial p_i}, \quad X_2 = \frac{1}{2}\sum_{i=1}^{n}\left(p_i\frac{\partial}{\partial p_i} - x_i\frac{\partial}{\partial x_i}\right), \quad X_3 = \sum_{i=1}^{n}\left(p_i\frac{\partial}{\partial x_i} + \frac{b_i}{x_i^3}\frac{\partial}{\partial p_i}\right),$$

fulfil the commutation rules (24). Hence, (35) is a Lie system. The space $\mathrm{T}^*\mathbb{R}_0^n$ admits a natural Poisson bivector $\Lambda = \sum_{i=1}^n \partial/\partial x_i \wedge \partial/\partial p_i$ related to the restriction to this space of the canonical symplectic structure on $\mathrm{T}^*\mathbb{R}^n$. Moreover, the preceding vector fields are Hamiltonian vector fields with Hamiltonian functions

$$h_1 = \frac{1}{2}\sum_{i=1}^{n} x_i^2, \qquad h_2 = -\frac{1}{2}\sum_{i=1}^{n} x_i p_i, \qquad h_3 = \frac{1}{2}\sum_{i=1}^{n}\left(p_i^2 + \frac{b_i}{x_i^2}\right) \tag{36}$$

which satisfy the commutation relations (25), so that $\mathcal{H}_\Lambda \simeq \mathfrak{sl}(2, \mathbb{R})$. Consequently, every curve h_t that takes values in the Lie algebra spanned by h_1, h_2 and h_3 gives rise to a Lie–Hamiltonian structure $(\mathrm{T}^*\mathbb{R}_0^n, \Lambda, h)$. Then, the system (35), described by the t-dependent vector field $X = X_3 + \omega^2(t)X_1 = -\widehat{\Lambda}(dh_3 + \omega^2(t)dh_1)$, is a Lie–Hamilton system with a Lie–Hamiltonian structure $(\mathrm{T}^*\mathbb{R}_0^n, \Lambda, h_t = h_3 + \omega^2(t)h_1)$.

Subsequently, we derive an explicit superposition rule for the simplest case of the system (35) corresponding to $n = 1$, and proceed as in the previous subsection. The prolongations of X_α ($\alpha = 1, 2, 3$) again become linearly independent for $m = 2$ and we need to obtain two first integrals for the diagonal prolongations $\widetilde{X}_\alpha$ of $\mathrm{T}^*\mathbb{R}_0^3$ fulfilling (31) for the coordinate system $\{x_{(1)}, p_{(1)}, x_{(2)}, p_{(2)}, x_{(3)}, p_{(3)}\}$ of $\mathrm{T}^*\mathbb{R}_0^3$. Similarly to the previous example, we have an injection $D : \mathfrak{sl}(2,\mathbb{R}) \to C^\infty(\mathrm{T}^*\mathbb{R}_0)$ which leads to the morphisms $D^{(2)}$ and $D^{(3)}$. Then, by taking into account the Casimir function (27) and the Hamiltonians (36), we apply Theorem 26 obtaining the following first integrals:

$$\begin{aligned} F^{(2)} = D^{(2)}(\Delta(C)) &= \tfrac{1}{4}(x_1p_2 - x_2p_1)^2 + \tfrac{b(x_1^2+x_2^2)^2}{4x_1^2x_2^2}, \\ F^{(3)} = D^{(3)}(\Delta(C)) &= \tfrac{1}{4}\sum_{1\le i<j}^{3}\left((x_ip_j - x_jp_i)^2 + \tfrac{b(x_i^2+x_j^2)^2}{x_i^2x_j^2}\right) - \tfrac{3}{4}b, \\ F_{13}^{(2)} = S_{13}(F^{(2)}), \qquad & F_{23}^{(2)} = S_{23}(F^{(2)}), \qquad F^{(3)} = F^{(2)} + F_{13}^{(2)} + F_{23}^{(2)} - 3b/4, \end{aligned}$$

where (x_i, p_i) denote the coordinates $(x_{(i)}, p_{(i)})$; notice that $F = D(C) = b/4$. We choose $F^{(2)}$ and $F_{23}^{(2)}$ as the two functionally independent constants of motion and we shall use $F_{13}^{(2)}$ in order to simplify the results. Recall that these functions are exactly the first integrals obtained in other works, e.g. [10], for describing superposition rules of dissipative Milne–Pinney equations (up to the diffeomorphism $\varphi : (x,p) \in \mathrm{T}^*\mathbb{R}_0 \mapsto (x,v) = (x,p) \in \mathrm{T}\mathbb{R}_0$ to system (35) with $n = 1$), and lead straightforwardly to deriving a superposition rule for these equations [60].

Indeed, we set

$$F^{(2)} = \frac{k_1}{4} + \frac{b}{2}, \qquad F_{23}^{(2)} = \frac{k_2}{4} + \frac{b}{2}, \qquad F_{13}^{(2)} = \frac{k_3}{4} + \frac{b}{2}, \tag{37}$$

and from the first equation we obtain p_1 in terms of the remaining variables and k_1:

$$p_1 = p_1(x_1, x_2, p_2, x_3, p_3, k_1) = \frac{p_2x_1^2x_2 \pm \sqrt{k_1x_1^2x_2^2 - b(x_1^4 + x_2^4)}}{x_1x_2^2}. \tag{38}$$

By introducing this value in the second expression of (37), one can determine the expression of x_1 as a function of x_2, p_2, x_3, p_3 and the constants k_1, k_2. Such a result is rather simplified when the third constant of (37) enters, yielding

$$\begin{aligned} x_1 &= x_1(x_2, p_2, x_3, p_3, k_1, k_2) = x_1(x_2, x_3, k_1, k_2, k_3) \\ &= \left[\mu_1x_2^2 + \mu_2x_3^2 \pm \sqrt{\mu\left[k_3x_2^2x_3^2 - b(x_2^4 + x_3^4)\right]}\right]^{1/2}, \end{aligned} \tag{39}$$

where the constants μ_1, μ_2, μ are defined in terms of k_1, k_2, k_3 and b as

$$\mu_1 = \frac{2bk_1 - k_2k_3}{4b^2 - k_3^2}, \qquad \mu_2 = \frac{2bk_2 - k_1k_3}{4b^2 - k_3^2}, \qquad \mu = \frac{4\left[4b^3 + k_1k_2k_3 - b(k_1^2 + k_2^2 + k_3^2)\right]}{(4b^2 - k_3^2)^2}.$$

And by introducing (39) into (38), we obtain $p_1 = p_1(x_2, p_2, x_3, p_3, k_1, k_2) = p_1(x_2, p_2, x_3, k_1, k_2, k_3)$, which together with (39) provide a superposition rule for (35) with $n = 1$. These expressions constitute the known superposition rule for Milne–Pinney equations [60]. Observe that, instead of solving systems of PDEs for obtaining the first integrals as in [10, 60], we have obtained them algebraically in a simpler way. When $b = 0$ we recover, as expected, the superposition rule for the harmonic oscillator with a t-dependent frequency. Similarly to previous examples, the above superposition rule is also valid when $\omega(t)$ is constant.

7.5 A classical system with trigonometric nonlinearities

Let us study a final example appearing in the study of integrability of classical systems [18, 54]. Consider the system

$$\left\{\begin{array}{ll}\frac{\mathrm{d}x}{\mathrm{d}t} & =\sqrt{1-x^2}\,(B_x(t)\sin\,p-B_y(t)\cos p)\,,\\ \frac{\mathrm{d}p}{\mathrm{d}t} & =-(B_x(t)\cos p+B_y(t)\sin p)\frac{x}{\sqrt{1-x^2}}-B_z(t),\end{array}\right.$$

where $B_x(t), B_y(t), B_z(t)$ are arbitrary t-dependent functions and $(x,p)\in \mathrm{T}^*\mathrm{I}$, with $\mathrm{I}=(-1,1)$. This system describes the integral curves of the t-dependent vector field

$$X=\sqrt{1-x^2}(B_x(t)\sin\,p-B_y(t)\cos p)\frac{\partial}{\partial x}-\left[\frac{(B_x(t)\cos p+B_y(t)\sin p)x}{\sqrt{1-x^2}}+B_z(t)\right]\frac{\partial}{\partial p},$$

which can be brought into the form $X=B_x(t)X_1+B_y(t)X_2+B_z(t)X_3$, where

$$X_1=\sqrt{1-x^2}\sin\,p\frac{\partial}{\partial x}-\frac{x}{\sqrt{1-x^2}}\cos p\frac{\partial}{\partial p},\;\; X_2=-\sqrt{1-x^2}\cos\,p\frac{\partial}{\partial x}-\frac{x}{\sqrt{1-x^2}}\sin p\frac{\partial}{\partial p},$$

and $X_3=-\partial/\partial p$ satisfy the commutation relations

$$[X_1,X_2]=X_3,\qquad [X_3,X_1]=X_2,\qquad [X_2,X_3]=X_1.$$

In other words, X describes a Lie system associated with a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{su}(2)$. As in the previous examples, we assume $V^X=V$. Now, the vector fields X_α $(\alpha=1,2,3)$ are Hamiltonian ones with Hamiltonian functions given by

$$h_1=-\sqrt{1-x^2}\cos p,\qquad h_2=-\sqrt{1-x^2}\sin p,\qquad h_3=x, \tag{40}$$

thus spanning a real Lie algebra isomorphic to $\mathfrak{su}(2)$. Indeed,

$$\{h_1,h_2\}=-h_3,\qquad \{h_3,h_1\}=-h_2,\qquad \{h_2,h_3\}=-h_1.$$

Next we consider a basis $\{v_1,v_2,v_3\}$ for $\mathfrak{su}(2)$ satisfying

$$[v_1,v_2]=-v_3,\qquad [v_3,v_1]=-v_2,\qquad [v_2,v_3]=-v_1,$$

so that $\mathfrak{su}(2)$ admits the Casimir $\mathcal{C}=v_1\widetilde{\otimes}v_1+v_2\widetilde{\otimes}v_2+v_3\widetilde{\otimes}v_3\in U_{\mathfrak{su}(2)}$. Then, the Casimir element of $S_{\mathfrak{su}(2)}$ reads $C=\lambda^{-1}(\mathcal{C})=v_1^2+v_2^2+v_3^2$.

The diagonal prolongations of X_1,X_2,X_3 are linearly independent at a generic point for $m=2$ and we have to derive two first integrals for the diagonal prolongations $\widetilde{X}_1,\widetilde{X}_2,\widetilde{X}_3$ on $\mathrm{T}^*\mathrm{I}^3$ satisfying (31) working with the coordinates $\{x_{(1)},p_{(1)},x_{(2)},p_{(2)},x_{(3)},p_{(3)}\}$ of $\mathrm{T}^*\mathrm{I}^3$. Then, by taking into account the Casimir function C, the Hamiltonians (40), the isomorphism $\phi(v_\alpha)=h_\alpha$ and the injection $D:\mathfrak{sl}(2,\mathbb{R})\to C^\infty(\mathrm{T}^*\mathrm{I})$, we apply Theorem 26 obtaining the following first integrals:

$$\begin{aligned} F^{(2)} &=2\left(\sqrt{1-x_1^2}\sqrt{1-x_2^2}\cos(p_1-p_2)+x_1x_2+1\right),\\ F^{(3)} &=2\textstyle\sum_{1\le i<j}^3\left(\sqrt{1-x_i^2}\sqrt{1-x_j^2}\cos(p_i-p_j)+x_ix_j\right)+3,\\ F^{(2)}_{13} &=S_{13}(F^{(2)}),\qquad F^{(2)}_{23}=S_{23}(F^{(2)}),\qquad F^{(3)}=F^{(2)}+F^{(2)}_{13}+F^{(2)}_{23}-3,\end{aligned}$$

and $F = D(C) = 1$. We again choose $F^{(2)}$ and $F^{(2)}_{23}$ as the two functionally independent constants of motion, which provide us, after cumbersome but straightforward computations, with a superposition rule for these systems. This leads to a quartic equation, whose solution can be obtained through known methods. All our results are also valid for the case when $V^X \subset V$.

8 Conclusions and outlook

We have proved several new properties of the constants of motion for Lie–Hamilton systems. New methods for their calculation have been devised, and Poisson coalgebra techniques have been developed for obtaining superposition rules.

Our achievements strongly simplify the search for constants of motion and superposition rules by avoiding the integration of ODEs and PDEs required by standard methods. We have provided generalisations of previous results on Lie–Hamilton systems [16, 20] and coalgebra integrability of autonomous Hamiltonian systems [19, 26]. Finally, we illustrated our approach by analysing several non-autonomous systems of interest.

In the future, we aim to apply our techniques to new relevant systems. Furthermore, we expect to extend our formalism to Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respect to a Dirac structure. This would enable us to use our procedures to study a broader variety of systems. Moreover, we also expect to analyse the use of Poisson coalgebra techniques to devise an algebraic approach to Lie–Hamilton systems with mixed superposition rules.

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Appendix

We now detail the proof of Lemma 17. The elements of $S_\mathfrak{g}$ given by $v^I \equiv v_1^{i_1} \cdot \ldots \cdot v_r^{i_r}$, where the I's are r-multi-indexes, "$\cdot$" denotes the product of elements of $\mathfrak{g}$ as functions on $\mathfrak{g}^*$ and $\{v_1, \ldots, v_r\}$ is a basis for $\mathfrak{g}$, form a basis of $S_\mathfrak{g}$. Then, every $P \in S_\mathfrak{g}$ can be written in a unique way as $P = \sum_{I \in M} \lambda_I v^I$, where M is a finite family of multi-indexes and each $\lambda_I \in \mathbb{R}$. Hence, the $\mathbb{R}$-algebra morphism $D : (S_\mathfrak{g}, \cdot) \to (C^\infty(N), \cdot)$ extending $\phi : \mathfrak{g} \to \mathcal{H}_\Lambda$ is determined by the image of the elements of a basis for $\mathfrak{g}$. Indeed,

$$D(P) = \sum_I \lambda_I D(v^I) = \sum_I \lambda_I \phi(v_1^{i_1}) \cdot \ldots \cdot \phi(v_r^{i_r}). \tag{41}$$

Let us prove that D is also an $\mathbb{R}$-algebra morphism. From (41), we see that D is linear. Moreover, $D(PQ) = D(P)D(Q)$ for every $P, Q \in S_\mathfrak{g}$. In fact, if we write $Q = \sum_{J\in M} \lambda_J v^J$, we obtain

$$D(PQ)=D\left(\sum_I \lambda_I v^I \sum_J \lambda_J v^J\right)=\sum_K \sum_{I+J=K} \lambda_I\lambda_J D(v^K)=\sum_I \lambda_I D(v^I)\sum_J \lambda_J D(v^J)=D(P)D(Q),$$

where $I + J = (i_1 + j_1, \ldots, i_r + j_r)$ with $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_r)$.

Let us show that D is also a Poisson morphism. By linearity, this reduces to proving that $D\left(\{v^I, v^J\}_{S_\mathfrak{g}}\right) = \{D(v^I), D(v^J)\}_\Lambda$ for arbitrary I and J. Define $|I| = i_1 + \ldots + i_r$. If $|J| = 0$ or $|I| = 0$ this is satisfied, as a Poisson bracket vanishes when any entry is a constant. We now prove by induction the remaining cases. For $|I| + |J| = 2$, we have

$$D\left(\{v_\alpha, v_\beta\}_{S_\mathfrak{g}}\right) = \phi([v_\alpha, v_\beta]_\mathfrak{g}) = \{\phi(v_\alpha), \phi(v_\beta)\}_\Lambda = \{D(v_\alpha), D(v_\beta)\}_\Lambda, \quad \forall \alpha, \beta = 1, \ldots, r.$$

If D is a Poisson morphism for $|I| + |J| = m > 2$, then for $|I| + |J| = m + 1$ we can set $v^I = v^{\bar{I}} v_\gamma^{i_\gamma}$ for $i_\gamma \neq 0$ and some γ to obtain

$$\begin{aligned} D\left(\{v^I, v^J\}_{S_\mathfrak{g}}\right) &= D\left(\{v^{\bar{I}} v_\gamma^{i_\gamma}, v^J\}_{S_\mathfrak{g}}\right) = D\left(\{v^{\bar{I}}, v^J\}_{S_\mathfrak{g}} v_\gamma^{i_\gamma} + v^{\bar{I}}\{v_\gamma^{i_\gamma}, v^J\}_{S_\mathfrak{g}}\right) \\ &= \{D(v^{\bar{I}}), D(v^J)\}_\Lambda D(v_\gamma^{i_\gamma}) + D(v^{\bar{I}})\{D(v_\gamma^{i_\gamma}), D(v^J)\}_\Lambda \\ &= \{D(v^{\bar{I}}) D(v_\gamma^{i_\gamma}), D(v^J)\}_\Lambda = \{D(v^I), D(v^J)\}_\Lambda. \end{aligned}$$

By induction, $D\left(\{v^I, v^J\}_{S_\mathfrak{g}}\right) = \{D(v^I), D(v^J)\}_\Lambda$ for any I and J.

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