Physics Letters A 375 (2011) 537–540

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Generalized Qiao hierarchy in $2+1$ dimensions: Reciprocal transformations, spectral problem and non-isospectrality

P.G. Estévez

Departamento de Física Fundamental, Área de Física Teórica, Universidad de Salamanca, Spain

ARTICLE INFO

Article history:
Received 15 October 2010
Received in revised form 2 December 2010
Accepted 12 December 2010
Available online 16 December 2010
Communicated by C.R. Doering

Keywords:
Recursion operators
Spectral problem
Reciprocal transformations

ABSTRACT

A new integrable $2+1$ hierarchy that generalizes the $1+1$ Qiao hierarchy is presented. A reciprocal transformation is defined such that the independent x-variable is considered as a dependent field. The hierarchy transforms into n copies of an equation that has a two component non-isospectral Lax pair. Coming back to the original variables, by inverting the reciprocal transformation we obtain a two component spectral problem for the $2+1$ hierarchy. Actually, this spectral problem is non-isospectral.

1. Introduction

- Recently Qiao and Liu [12] have proposed the following integrable equation:

$$u_t = \left(\frac{1}{2u^2}\right)_{xxx} - \left(\frac{1}{2u^2}\right)_x, \tag{1}$$

which has peaked solutions. More research about the solutions of this equation can be found in [13]. Furthermore, this equation has a bihamiltonian structure [12]:

$$u_t = J\frac{\delta H_1}{\delta u} = K\frac{\delta H_2}{\delta u}, \tag{2}$$

where the operators K and J are:

$$J = -\partial u \partial^{-1} u \partial, \qquad \partial = \frac{\partial}{\partial x}, \tag{3}$$

$$K = \partial^3 - \partial \tag{4}$$

this allows us to define the recursion operator:

$$R = JK^{-1}. \tag{5}$$

This recursion operator was used by Qiao in [11] to construct a $1+1$ integrable hierarchy. Eq. (1) is the second positive member of the Qiao hierarchy. The second negative member of the hierarchy was investigated by the same author in [10].

- In [12], the authors compare (1) with the Harry–Dym case [15], which reads

$$u_t = \left(\frac{1}{\sqrt{u}}\right)_{xxx} - \left(\frac{1}{\sqrt{u}}\right)_x. \tag{6}$$

(6) is also known as the first member of the positive flow of the Camassa–Holm hierarchy [4,7,9]. In contrast the celebrated Camassa–Holm equation [1] is the first member of the negative flow. Both flows of the Camassa–Holm hierarchy have been extensively studied in [9].
- In [4], an integrable generalization to $2+1$ dimensions of the Camassa–Holm hierarchy was presented. By using reciprocal transformations, the spectral problem for such a hierarchy was obtained. The spectral problem was in fact non-isospectral.

With this in mind, our aim in this Letter is to use the recursion operator (5) to generate a $2+1$ hierarchy in a way similar to that used in [4]. The plan of the Letter is as follows:

- In Section 2 we shall introduce the new hierarchy and we shall write it explicitly for the n-component. This hierarchy will be named in the following GQH (generalized Qiao hierarchy). A reciprocal transformation [6,7,4] can be defined in which the independent variable x will be the new dependent field. This reciprocal transformation will be extended by transforming the dependent fields of the hierarchy into new independent variables. The nice result is the transformation of n-component of the GQH hierarchy into n copies of an equation in just three variables. This equation happens to be precisely the equation presented in [3] as a generalization of the sine-Gordon equation.

E-mail address: pilar@usal.es.

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2010.12.021

- The result of Section 2 allows us to use the results of [3] to obtain the Lax pair of each of the n equations obtained from the reciprocal transformation of our original hierarchy. When we write these Lax pair in terms of the original variables and fields, we shall obtain the spectral problem associated with the GQH hierarchy. This spectral problem is in fact non-isospectral.
- We close the Letter with a section of conclusions.

2. A new hierarchy in 2 + 1 dimensions

By analogy with the results of [4], we can use the recursion operator defined in (5) to construct the 2 + 1 GQH hierarchy.

$$u_t = R^{-n} u_y. \tag{7}$$

The n component of this hierarchy can be written explicitly by introducing n fields $v^{[j]}$, defined as:

$$J v^{[j+1]} = K v^{[j]} \quad \Longrightarrow \quad v^{[j+1]} = J^{-1} K v^{[j]}, \quad j = 1 \dots n-1. \tag{8}$$

Eq. (7) can be written now as:

$$u_y = J v^{[1]}, \tag{9}$$

$$u_t = K v^{[n]}. \tag{10}$$

It is also useful to introduce the following functions $\omega^{[j]}$:

$$\omega^{[j]} = \partial^{-1} u \partial v^{[j]} \quad \Longrightarrow \quad \omega_x^{[j]} = u v_x^{[j]}, \quad j = 1 \dots n, \tag{11}$$

which allows us to write (8)–(10) as:

$$v_{xx}^{[j]} - v^{[j]} = -u\,\omega^{[j+1]}, \quad j = 1 \dots n-1, \tag{12}$$

$$u_t = \left(v_{xx}^{[n]} - v^{[n]}\right)_x, \tag{13}$$

$$u_y = -\left(u\omega^{[1]}\right)_x. \tag{14}$$

2.1. Reduction to the Qiao–Liu equation

Note that in the case $n = 1$, if the fields do not depend on the y variable the above equations reads:

$$\omega_x^{[1]} = u\, v_x^{[1]},$$

$$u_t = \left(v_{xx}^{[1]} - v^{[1]}\right)_x,$$

$$\left(u\,\omega^{[1]}\right)_x = 0$$

which can easily be written as

$$\omega^{[1]} = \frac{1}{u}, \qquad v^{[1]} = \frac{1}{2u^2},$$

$$u_t = \left(\frac{1}{2u^2}\right)_{xxx} - \left(\frac{1}{2u^2}\right)_x, \tag{15}$$

that is the Qiao–Liu equation.

2.2. Reciprocal transformation

The conservative form of Eqs. (13) and (14) suggests the introduction of the following exact derivative:

$$u\,dx - u\omega^{[1]}\,dy + \left(v_{xx}^{[n]} - v^{[n]}\right)dt.$$

We can now introduce a reciprocal transformation [6,7,14] such that the new independent variables are Z_0, Z_1, Z_{n+1}, defined as:

$$dZ_0 = u\,dx - u\,\omega^{[1]}\,dy + \left(v_{xx}^{[n]} - v^{[n]}\right)dt,$$

$$Z_1 = y,$$

$$Z_{n+1} = t, \tag{16}$$

and the variable x is now considered as a new dependent field H field [5]

$$x = H(Z_0, Z_1, Z_{n+1}). \tag{17}$$

From (16), we have:

$$dx = \frac{1}{u}\,dZ_0 + \omega^{[1]}\,dZ_1 - \frac{(v_{xx}^{[n]} - v^{[n]})}{u}\,dZ_{n+1}, \tag{18}$$

which means that:

$$H_0 = \frac{1}{u} \quad \Longrightarrow \quad u = \frac{1}{H_0}, \tag{19}$$

$$H_{n+1} = -\frac{(v_{xx}^{[n]} - v^{[n]})}{u} \quad \Longrightarrow \quad \left(v_{xx}^{[n]} - v^{[n]}\right) = -\frac{H_{n+1}}{H_0}, \tag{20}$$

$$H_1 = \omega^{[1]} \tag{21}$$

where we have used the notation:

$$H_i = \frac{\partial H}{\partial Z_i}.$$

2.3. Extension of the reciprocal transformation

Following the procedure introduced in [4], we can extend the number of independent variables to a set $(Z_0, Z_1, Z_2, \dots, Z_n, Z_{n+1})$, where the new independent variables are introduced as an extension of (21) in the form:

$$H_j = \omega^{[j]}, \quad j = 1 \dots n. \tag{22}$$

Therefore we have a new independent variable Z_j for each of the n former dependent fields $\omega^{[j]}$. This reciprocal transformation obviously implies that the field H is actually $H = H(Z_0, Z_1, Z_2, \dots, Z_n, Z_{n+1})$ and therefore it depends on $n + 2$ independent variables.

2.4. Transformation of the derivatives

It is easy to see that the above defined reciprocal transformation yields the following transformation of the partial derivatives:

$$\partial_x = u\partial_0 = \frac{1}{H_0}\partial_0,$$

$$\partial_y = \partial_1 - u\omega^{[1]}\partial_0 = \partial_1 - \frac{H_1}{H_0}\partial_0,$$

$$\partial_t = \partial_{n+1} + \left(v_{xx}^{[n]} - v^{[n]}\right)\partial_0 = \partial_{n+1} - \frac{H_{n+1}}{H_0}\partial_0. \tag{23}$$

2.5. Transformation of the equations

Let us go to apply the reciprocal transformation to the GQH hierarchy (11)–(14):

- Eq. (14) is trivially satisfied, and by applying the transformation to Eq. (11), we have:

$$\frac{1}{H_0} H_{0j} = \frac{1}{H_0^2}\partial_0\left(v^{[j]}\right) \to \partial_0\left(v^{[j]}\right) = \partial_j\left(\frac{H_0^2}{2}\right), \quad j = 1 \dots n. \tag{24}$$

The conservative form of this latter equation, suggests the introduction of a new field $R(Z_0, Z_1, \dots, Z_n Z_{n+1})$, such that we have:

$$R_0 = \frac{H_0^2}{2}, \tag{25}$$

$$v^{[j]} = R_j, \quad j = 1 \dots n. \tag{26}$$

- The transformation of Eqs. (12) and (13) yields:

$$R_j = \frac{H_{j00}+H_{j+1}}{H_0}, \quad j=1\dots n-1, \tag{27}$$

$$R_n = \frac{H_{n00}+H_{n+1}}{H_0}. \tag{28}$$

Summarizing: the reciprocal transformation (16)–(22) yield n systems of the form

$$R_0 = \frac{H_0^2}{2},$$
$$R_j = \frac{H_{j00}+H_{j+1}}{H_0}, \quad j=1\dots n, \tag{29}$$

where each system depends on just three variables: Z_0, Z_j, Z_{j+1}. System (29) was derived by Kudryashov and Pickering [8] as a member of a $(2+1)$ Schwarzian breaking soliton hierarchy. Dromions and other soliton solution of (29) have been obtained in [3]. System (29) can be considered as a modified version of the CBS (Calogero–Bogoyavlenski–Schiff) equation. This CBS equation, when written in the Z_0, Z_j, Z_{j+1} variables, is:

$$M_{0,j+1}+M_{000j}+4M_jM_{00}+8M_0M_{0j}=0. \tag{30}$$

In [3] it was proved that (29) and (30) are related by means of the following Miura transformation:

$$M_0 = -\frac{H_0^2}{8} \pm \frac{H_{00}}{4},$$
$$M_j = \frac{-H_{j+1}\pm 4M_{0j}}{4H_0}, \quad j=1\dots n. \tag{31}$$

In the next section we shall use the results of [3] to construct a Lax pair for the GQH hierarchy.

3. Spectral problem

3.1. Spectral problem for (29)

In [3] the Miura transformation (31) was used to derive a two component Lax pair for the system (29). In our variables this spectral problem reads:

$$\psi_0 = i\frac{\sqrt{\lambda}}{2}\phi - \frac{H_0}{2}\psi, \tag{32}$$

$$\phi_0 = i\frac{\sqrt{\lambda}}{2}\psi + \frac{H_0}{2}\phi, \tag{33}$$

$$\psi_{j+1} = \lambda\psi_j + i\frac{\sqrt{\lambda}}{2}(R_j - H_{0j})\phi - \frac{H_{j+1}}{2}\psi, \quad j=1\dots n, \tag{34}$$

$$\phi_{j+1} = \lambda\phi_j + i\frac{\sqrt{\lambda}}{2}(R_j + H_{0j})\psi + \frac{H_{j+1}}{2}\phi, \quad j=1\dots n. \tag{35}$$

It is easy to see that the compatibility condition of (32)–(35) yields the system (29) as well as the following non-isospectral condition:

$$\lambda_0 = 0,$$
$$\lambda_{j+1} = \lambda\lambda_j. \tag{36}$$

3.2. Spectral problem for (11)–(14)

If, from the above Lax pair, we wish to obtain the spectral problem of the GQH hierarchy, we need to invert the reciprocal transformation (19)–(22), which means applying the following substitutions:

$$H_0 = \frac{1}{u}, \tag{37}$$

$$H_j = \omega^{[j]}, \quad j=1\dots n, \tag{38}$$

$$H_{n+1} = -\frac{v_{xx}^{[n]} - v^{[n]}}{u}, \tag{39}$$

$$R_0 = \frac{1}{2u^2}, \tag{40}$$

$$R_j = V^{[j]}. \tag{41}$$

The inverse of (23) is:

$$\partial_0 = \frac{1}{u}\partial_x,$$
$$\partial_1 = \partial_y + \omega^{[1]}\partial_x,$$
$$\partial_{n+1} = \partial_t - \frac{(v_{xx}^{[n]} - v^{[n]})}{u}\partial_x. \tag{42}$$

It is useful to notice that by using (11) we have:

$$H_{j0} = \frac{1}{u}\partial_x(\omega^{[j]}) = v_x^{[j]}. \tag{43}$$

We can now tackle the transformation of (32)–(35). The two first equation transform trivially to:

$$\psi_x = i\frac{\sqrt{\lambda}}{2}u\phi - \frac{\psi}{2}, \tag{44}$$

$$\phi_x = i\frac{\sqrt{\lambda}}{2}u\psi + \frac{\phi}{2}. \tag{45}$$

The transformation of (34)–(35) is slightly more complicated. Let us write them in the following form:

$$E_j = \psi_{j+1} - \lambda\psi_j - i\frac{\sqrt{\lambda}}{2}(R_j - H_{0j})\phi + \frac{H_{j+1}}{2}\psi = 0, \quad j=1\dots n,$$
$$F_j = \phi_{j+1} - \lambda\phi_j - i\frac{\sqrt{\lambda}}{2}(R_j + H_{0j})\psi - \frac{H_{j+1}}{2}\phi = 0, \tag{46}$$

and by performing the sums $\sum_{j=1}^{n}\lambda^{n-j}E_j$ and $\sum_{j=1}^{n}\lambda^{n-j}F_j$ and with the aid of (37)–(42), we have that, in the original variables, the following equations hold:

$$\psi_t = \lambda^n\psi_y + \lambda\sum_{j=1}^{n}\lambda^{n-j}\omega^{[j]}\psi_x + i\frac{\sqrt{\lambda}}{2}\sum_{j=1}^{n}\lambda^{n-j}\left(v_{xx}^{[j]} - v^{[j]}\right)\phi, \tag{47}$$

$$\phi_t = \lambda^n\phi_y + \lambda\sum_{j=1}^{n}\lambda^{n-j}\omega^{[j]}\phi_x + i\frac{\sqrt{\lambda}}{2}\sum_{j=1}^{n}\lambda^{n-j}\left(v_{xx}^{[j]} + v^{[j]}\right)\psi. \tag{48}$$

The two component Lax pair can be written in a more compact form as:

$$\begin{pmatrix}\psi\\ \phi\end{pmatrix}_x = \begin{pmatrix}-\frac{1}{2} & i\frac{\sqrt{\lambda}}{2}u\\ i\frac{\sqrt{\lambda}}{2}u & \frac{1}{2}\end{pmatrix}\begin{pmatrix}\psi\\ \phi\end{pmatrix}, \tag{49}$$

$$\begin{pmatrix}\psi\\ \phi\end{pmatrix}_t = \lambda^n\begin{pmatrix}\psi\\ \phi\end{pmatrix}_y + \lambda\hat{A}\begin{pmatrix}\psi\\ \phi\end{pmatrix}_x + i\frac{\sqrt{\lambda}}{2}\begin{pmatrix}0 & \hat{B}_{xx}-\hat{B}\\ \hat{B}_{xx}+\hat{B} & 0\end{pmatrix}\begin{pmatrix}\psi\\ \phi\end{pmatrix} \tag{50}$$

where

$$\hat{A}=\sum_{j=1}^{n}\lambda^{n-j}\omega^{[j]},\qquad \hat{B}=\sum_{j=1}^{n}\lambda^{n-j}v^{[j]}. \tag{51}$$

And finally the non-isospectral conditions (36) can be written as:

$$\lambda_0=0,\qquad \sum_{j=1}^{n}\lambda^{n-j}\lambda_{j+1}-\sum_{j=1}^{n}\lambda^{n-j}\lambda\lambda_j=\lambda_{n+1}-\lambda^n\lambda_1=0.$$

If we apply the inverse transformation (43), we have:

$$\lambda_x=0,\qquad \lambda_t-\lambda^n\lambda_y=0. \tag{52}$$

It is easy to check that if the non-isospectral condition (52) is satisfied then the compatibility condition of (3.19) and (3.20) yields the set of Eqs. (2.5)–(2.8). Therefore, (3.18)–(3.19) are a Lax pair for the GQH hierarchy. This same behavior for the spectral parameter is exhibited by the 2 + 1 Camassa–Holm hierarchy [4]. Different non-isospectral scattering problems are described in [2].

4. Conclusions

- We have presented a new integrable hierarchy in 2 + 1 dimensions that is a generalization of the Qiao–Liu equation in 1 + 1 dimensions. The n-component of this hierarchy is also written explicitly by introducing n auxiliary fields $v^{[j]}$.
- A reciprocal transformation is defined such that the former x-variable is the new field and we extend the transformation by introducing a new dependent variable, Z_j, for each of the fields $v^{[j]}$. The result is a field $x=H(Z_0\ldots Z_{n+1})$ depending on $n+2$ variables.
- With this reciprocal transformation, the GQH hierarchy transforms to a set of n different copies of the same equation. Each copy depends on just three variables. This equation is precisely the one studied in [3], where a two component non-isospectral Lax pair was constructed.
- The results of [3] allow us, by inversion of the reciprocal transformation, to obtain a non-isospectral Lax pair for the GQH hierarchy.

Acknowledgements

This research has been supported in part by the DGICYT under project FIS2009-07880 and JCyL under contract GR224.

References

[1] R. Camassa, D.D. Holm, Phys. Rev. Lett. 71 (1993) 1661.
[2] P.A. Clarkson, P.R. Gordoa, A. Pickering, Inverse Problems 13 (1997) 1463.
[3] P.G. Estévez, J. Prada, J. Nonlin. Math. Phys. 11 (2004) 164.
[4] P.G. Estévez, J. Prada, J. Phys. A: Math. Gen. 38 (2005) 1287.
[5] P.G. Estévez, Theor. Math. Phys. 159 (2009) 763.
[6] N. Euler, M.L. Gandarias, M. Euler, O. Lindblom, J. Math. Anal. Appl. 257 (2001) 21.
[7] A.N.W. Hone, J.P. Ping, Inverse Problems 19 (2003) 129.
[8] N. Kudryashov, A. Pickering, J. Phys. A: Math. Gen. 31 (1998) 9505.
[9] Z. Qiao, Commun. Math. Phys. 239 (2003) 309.
[10] Z. Qiao, J. Math. Phys. 47 (2006) 112701 (9 pages).
[11] Z. Qiao, J. Math. Phys. 48 (2007) 082701 (20 pages).
[12] Z. Qiao, L. Liu, Chaos Solitons Fractals 41 (2009) 587.
[13] Z. Qiao, J. Math. Phys. 51 (2010) 042703 (23 pages).
[14] C. Rogers, M.C. Nucci, Phys. Scripta 33 (1986) 55.
[15] M. Wadati, K. Konno, Y.H. Ichikawa, J. Phys. Soc. Japan 47 (1979) 1698.