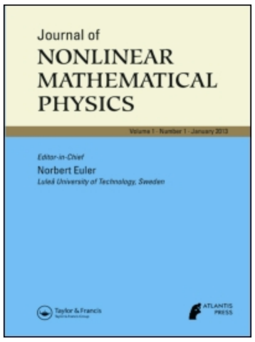

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Journal of Nonlinear Mathematical Physics

ISSN: 1402-9251 (Print) 1776-0852 (Online) Journal homepage: http://www.tandfonline.com/loi/tnmp20

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To cite this article: P. Albares, J. M. Conde & P. G. Estévez (2017) Classical Lie symmetries and reductions for a generalized NLS equation in 2+1 dimensions, Journal of Nonlinear Mathematical Physics, 24:sup1, 48-60, DOI: 10.1080/14029251.2017.1418053

To link to this article: https://doi.org/10.1080/14029251.2017.1418053

Journal of Nonlinear Mathematical Physics, Vol. 24, Supplement 1 (2017) 48–60

Classical Lie symmetries and reductions for a generalized NLS equation in $2+1$ dimensions

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Received 26 July 2017

Accepted 5 September 2017

A non-isospectral linear problem for an integrable $2+1$ generalization of the non linear Schrödinger equation, which includes dispersive terms of third and fourth order, is presented. The classical symmetries of the Lax pair and the related reductions are carefully studied. We obtain several reductions of the Lax pair that yield in some cases non-isospectral problems in $1+1$ dimensions.

Keywords: Lie symmetries, similarity reductions, Lax pair.

2000 Mathematics Subject Classification: 35C06, 35C30, 35051

1. Introduction

Recently, an integrable system [6] in $2+1$ dimensions has been proposed as a promising starting model for describing energy transfer processes in α-helical proteins within the continuum limit. This model is described by the following system of equations:

$$\begin{aligned}
&iu_t + u_{xy} + 2um_y + i\gamma_2\left(u_{xxx} - 6u\omega u_x\right)\\
&\qquad + \gamma_1\left(u_{xxxx} - 8u\omega u_{xx} - 2u^2\omega_{xx} - 4uu_x\omega_x - 6\omega u_x^2 + 6u^3\omega^2\right) = 0\,,\\
&-i\omega_t + \omega_{xy} + 2\omega m_y - i\gamma_2\left(\omega_{xxx} - 6u\omega\omega_x\right)\\
&\qquad + \gamma_1\left(\omega_{xxxx} - 8u\omega\omega_{xx} - 2\omega^2 u_{xx} - 4\omega u_x\omega_x - 6u\omega_x^2 + 6u^2\omega^3\right) = 0\,,\\
&(m_x + u\omega)_y = 0\,,
\end{aligned} \tag{1.1}$$

where m is a real field and $\omega = u^*$. This equation can be considered as a higher order nonlinear Schrödinger equation (HONLS in the future) that includes third ($\gamma_2 \neq 0$) and fourth ($\gamma_1 \neq 0$) order derivatives with respect to x. A non-isospectral Lax pair for HONLS was obtained in [6]. It has the following form:

$$
\begin{aligned}
&\psi_x = -i\lambda\,\psi - \chi\, u,\\
&\chi_x = i\chi\,\lambda - \psi\,\omega,\\
&\psi_t = -i\gamma_2(-2\,i\chi\,u^2\omega - 4\,i\chi\,u\lambda^2 - i\psi\,u\omega_x + i\psi\,\omega u_x + 2\,\psi\,u\omega\lambda +\\
&\quad 4\,\psi\,\lambda^3 + 2\,\chi\,\lambda u_x + i\chi\,u_{xx}) + i\gamma_1(-4\,i\chi\,u^2\omega\lambda - 8\,i\chi\,u\lambda^3 - 2\,i\psi\,u\lambda\,\omega_x +\\
&\quad 2\,i\psi\,\omega\lambda u_x + 3\,\psi\,u^2\omega^2 + 4\,\psi\,u\omega\lambda^2 + 8\,\psi\,\lambda^4 + 2\,i\chi\,\lambda u_{xx} + 6\,\chi\,u\omega u_x +\\
&\quad 4\,\chi\,\lambda^2 u_x - \psi\,u\omega_{xx} - \psi\,\omega u_{xx} + \psi\,u_x\omega_x - \chi\,u_{xxx}) + 2\,\lambda\,\psi_y + i\psi\,m_y - i\chi\,u_y + \psi\lambda_y,\\
&\chi_t = i\gamma_2(2\,i\psi\,u\omega^2 + 4\,i\psi\,\omega\lambda^2 + 2\,\chi\,u\omega\lambda + 4\,\chi\,\lambda^3 - i\psi\,\omega_{xx} -\\
&\quad i\chi\,u\omega_x + i\chi\,\omega u_x + 2\,\psi\,\lambda\,\omega_x) + i\gamma_1(-4\,i\psi\,u\omega^2\lambda - 8\,i\psi\,\omega\lambda^3 + 2\,i\chi\,u\lambda\,\omega_x -\\
&\quad 2\,i\chi\,\omega\lambda u_x - 3\,\chi\,u^2\omega^2 - 4\,\chi\,u\omega\lambda^2 - 8\,\chi\,\lambda^4 + 2\,i\psi\,\lambda\,\omega_{xx} - 6\,\psi\,u\omega\omega_x -\\
&\quad 4\,\psi\,\lambda^2\omega_x + \chi\,u\omega_{xx} + \chi\,\omega u_{xx} - \chi\,u_x\omega_x + \psi\,\omega_{xxx}) + 2\,\lambda\,\psi_y - i\psi\,m_y + i\chi\,\omega_y + \chi\lambda_y,
\end{aligned}
\tag{1.2}
$$

where $\psi(x,y,t)$ and $\chi(x,y,t)$ are the eigenfunctions, i is the imaginary unit ($i^2 = -1$), and $\lambda(y,t)$ is the spectral parameter. Furthermore, the equations for ψ are the complex conjugate of the equations for χ. The compatibility condition of the cross-derivatives yields (1.1) as well as the non-isospectral condition

$$\lambda_t - 2\lambda\lambda_y = 0 \tag{1.3}$$

The system (1.1) generalizes to $2+1$ dimensions the system proposed by Ankiewitz *et al* in [1]. This equation of reference [1] contains many integrable particular cases such as the standard NLS equation ($\gamma_1 = \gamma_2 = 0$), the Hirota equation ($\gamma_1 = 0$) [8] and the Lakshmanan-Porsezian- Daniel equation ($\gamma_2 = 0$) [9]. Furthermore (1.1), when $\gamma_1 = \gamma_2 = 0$, reduces to a $2+1$ NLS equation [2] that has been extensively analyzed in [3].

The classical Lie approach is a very well established procedure [12], [13] to get point symmetries of a system of differential equations. Nevertheless, in this paper we are more concerned with the identification of symmetries of the Lax pair (1.2). This approach [4], [10] has the benefit that the reduction associated to each symmetry of the Lax pair provides, not only the reduction of the fields u, ω and m, but the reductions of the eigenfuntions and the spectral parameter itself [5], [7].

We shall apply, in section 2, the classical Lie procedure to identify the symmetries of (1.2) in the four cases that arise from the different combinations of the possible values of γ_1 and γ_2. Sections 3, 4, 5 and 6 are devoted to consider the reductions associated to the symmetries identified in section 2 for the above mentioned four different cases. We close with a section of conclusions.

Co-published by Atlantis Press and Taylor & Francis

2. Classical Lie symmetries

In this section we apply the classical Lie approach [11], [14], in order to obtain symmetries of (1.2). Let us consider the following infinitesimal transformation [15]

$$\begin{aligned}
&x \to x+\varepsilon\xi_1(x,y,t,u,\omega,m,\lambda,\psi,\chi),\\
&y \to y+\varepsilon\xi_2(x,y,t,u,\omega,m,\lambda,\psi,\chi),\\
&t \to t+\varepsilon\xi_3(x,y,t,u,\omega,m,\lambda,\psi,\chi),\\
&u \to u+\varepsilon\eta_1(x,y,t,u,\omega,m,\lambda,\psi,\chi),\\
&\omega \to \omega+\varepsilon\eta_2(x,y,t,u,\omega,m,\lambda,\psi,\chi),\\
&m \to m+\varepsilon\eta_3(x,y,t,u,\omega,m,\lambda,\psi,\chi),\\
&\lambda \to \lambda+\varepsilon\eta_4(x,y,t,u,\omega,m,\lambda,\psi,\chi),\\
&\psi \to \psi+\varepsilon\phi_1(x,y,t,u,\omega,m,\lambda,\psi,\chi),\\
&\chi \to \chi+\varepsilon\phi_2(x,y,t,u,\omega,m,\lambda,\psi,\chi),
\end{aligned} \tag{2.1}$$

where ε is the parameter of the Lie group and ξ_1, ξ_2, ξ_3, η_1, η_2, η_3, η_4, ϕ_1 and ϕ_2 are the infinitesimals of the vector field

$$X=\xi_1\frac{\partial}{\partial x}+\xi_2\frac{\partial}{\partial y}+\xi_3\frac{\partial}{\partial t}+\eta_1\frac{\partial}{\partial u}+\eta_2\frac{\partial}{\partial \omega}+\eta_3\frac{\partial}{\partial m}+\eta_4\frac{\partial}{\partial \lambda}+\phi_1\frac{\partial}{\partial \psi}+\phi_2\frac{\partial}{\partial \chi}. \tag{2.2}$$

This infinitesimal transformation induces a well known one in the derivatives of the fields [13] [15]. This procedure, when applied to (1.2), yields an overdetermined system o PDEs, whose solution provides the infinitesimals. We shall remark that Lie approach requires a substitution of the higher order derivatives. The order of the higher derivatives in (1.2) is different depending whether γ_1 and γ_2 are (one or both) equal or different from 0. It means that it is necessary to split the problem in four different cases depending on the different combinations of γ_1 and γ_2.

As we said in the introduction, we are dealing with symmetries and reductions of (1.2). These symmetries obviously provide the corresponding symmetries and reductions for (1.1). When the infinitesimals (2.1) are known, we can proceed to calculate the associated reductions by solving the following characteristic equation

$$\frac{dx}{\xi_1}=\frac{dy}{\xi_2}=\frac{dt}{\xi_3}=\frac{du}{\eta_1}=\frac{d\omega}{\eta_2}=\frac{dm}{\eta_3}=\frac{d\lambda}{\eta_4}=\frac{d\psi}{\phi_1}=\frac{d\chi}{\phi_2} \tag{2.3}$$

2.1. *The case* $\gamma_1 \neq 0$ *and* $\gamma_2 \neq 0$

By applying the infinitesimal transformation (2.1) to (1.2) and, following the standard procedure [13], [15], we obtain the infinitesimals listed in Table 1.

$\xi_1 = K_1(t)$	$\xi_2 = \alpha_1$	$\xi_3 = \alpha_2$
$\eta_1 = iu\left(\dot{K}_1(t)y + 2K_2(t)\right)$	$\eta_2 = -iw\left(\dot{K}_1(t)y + 2K_2(t)\right)$	
$\eta_3 = \frac{1}{4}\ddot{K}_1(t)y^2 + \dot{K}_2(t)y + K_3(x,t)$	$\eta_4 = 0$	
$\phi_1 = \psi\left(\frac{i}{2}\dot{K}_1(t)y + iK_2(t) + K_0(y,t,\lambda)\right)$	$\phi_2 = \chi\left(-\frac{i}{2}\dot{K}_1(t)y - iK_2(t) + K_0(y,t,\lambda)\right)$	

Table 1. $\gamma_1 \neq 0$, $\gamma_2 \neq 0$

The dot denotes derivative with respect to t. α_1 and α_2 are arbitrary constants and K_i $(i = 1,...,3)$ are arbitrary functions of the indicated variables. Furthermore $K_0(y,t,\lambda)$ should satisfy the equation.

$$\frac{\partial K_0(y,t,\lambda)}{\partial t} - 2\lambda\frac{\partial K_0(y,t,\lambda)}{\partial y} = 0. \tag{2.4}$$

2.2. *The case* $\gamma_1 \neq 0$ *and* $\gamma_2 = 0$

In this section we are considering (1.2), when $\gamma_2 = 0$. If we apply the infinitesimal transformation (2.1), then we obtain the following infinitesimals

$\xi_1 = K_1(t) + \alpha_3 x$	$\xi_2 = \alpha_1 + 3\alpha_3 y$	$\xi_3 = \alpha_2 + 4\alpha_3 t$
$\eta_1 = iu\left(\dot{K}_1(t)y + 2K_2(t)\right) - \alpha_3 u$	$\eta_2 = -i\omega\left(\dot{K}_1(t)y + 2K_2(t)\right) - \alpha_3\omega$	
$\eta_3 = \frac{1}{4}\ddot{K}_1(t)y^2 + \dot{K}_2(t)y + K_3(x,t) - \alpha_3 m$	$\eta_4 = -\alpha_3\lambda$	
$\phi_1 = \psi\left(\frac{i}{2}\dot{K}_1(t)y + iK_2(t) + K_0(y,t,\lambda)\right)$	$\phi_2 = \chi\left(-\frac{i}{2}\dot{K}_1(t)y - iK_2(t) + K_0(y,t,\lambda)\right)$	

Table 2. $\gamma_1 \neq 0$, $\gamma_2 = 0$

where α_i $(i = 1,...,3)$, are arbitrary constants and K_j $(j = 1,...,3)$, are arbitrary functions of the indicated variables. $K_0(y,t,\lambda)$ satisfies the equation (2.4).

Notice that the infinitesimals are just the same as in Table 1, except for those terms related to the constant α_3.

2.3. *The case* $\gamma_1 = 0$ *and* $\gamma_2 \neq 0$

The procedure [13] yields the infinitesimals listed below (Table 3).

$\xi_1 = K_1(t) + \alpha_3 x$	$\xi_2 = \alpha_1 + 2\alpha_3 y$	$\xi_3 = \alpha_2 + 3\alpha_3 t$
$\eta_1 = iu\left(\dot{K}_1(t)y + 2K_2(t)\right) - \alpha_3 u$	$\eta_2 = -iw\left(\dot{K}_1(t)y + 2K_2(t)\right) - \alpha_3\omega$	
$\eta_3 = \frac{1}{4}\ddot{K}_1(t)y^2 + \dot{K}_2(t)y + K_3(x,t) - \alpha_3 m$	$\eta_4 = -\alpha_3\lambda$	
$\phi_1 = \psi\left(\frac{i}{2}\dot{K}_1(t)y + iK_2(t) + K_0(y,t,\lambda)\right)$	$\phi_2 = \chi\left(-\frac{i}{2}\dot{K}_1(t)y - iK_2(t) + K_0(y,t,\lambda)\right)$	

Table 3. $\gamma_1 = 0$, $\gamma_2 \neq 0$

α_i $(i = 1,...,3)$ are arbitrary constants and K_j $(j = 1,...,3)$ are arbitrary functions of the indicated variables. $K_0(y,t,\lambda)$ satisfies the equation (2.4).

We can easily see that the only additional symmetry to those listed in Table 3 is the symmetry associated to α_3.

2.4. *The case* $\gamma_1 = 0$ *and* $\gamma_2 = 0$

The infinitesimals obtained in this case are presented in Table 4.

$\xi_1 = K_1(t) + \alpha_3 x + 2\alpha_5 xt$	$\xi_2 = \alpha_1 + \alpha_4 y + 2\alpha_6 t + 2\alpha_5 yt$	$\xi_3 = \alpha_2 + \alpha_3 t + \alpha_4 t + 2\alpha_5 t^2$
$\eta_1 = iu\left(\dot{K}_1(t)y + 2K_2(t) + 2x(\alpha_5 y + \alpha_6)\right) - (\alpha_3 + 2\alpha_5 t)\, u$	$\eta_2 = -i\omega\left(\dot{K}_1(t)y + 2K_2(t) + 2x(\alpha_5 y + \alpha_6)\right) - (\alpha_3 + 2\alpha_5 t)\, \omega$	
$\eta_3 = \frac{1}{4}\ddot{K}_1(t)y^2 + \dot{K}_2(t)y + K_3(x,t) - (\alpha_3 + 2\alpha_5 t)\, m$	$\eta_4 = -\alpha_3 \lambda - \alpha_5(2t\lambda + y) - \alpha_6$	
$\phi_1 = \psi\left(\frac{i}{2}\dot{K}_1(t)y + iK_2(t) + K_0(y,t,\lambda)\right) + \psi(ix(\alpha_5 y + \alpha_6) - \alpha_5 t)$	$\phi_2 = \chi\left(-\frac{i}{2}\dot{K}_1(t)y - iK_2(t) + K_0(y,t,\lambda)\right) + \chi(-ix(\alpha_5 y + \alpha_6) - \alpha_5 t)$	

Table 4. $\gamma_1 = 0$, $\gamma_2 = 0$

where α_i $(i = 1,...,6)$ are arbitrary constants and K_j $(j = 1,...,3)$, are arbitrary functions of the indicated variables. $K_0(y,t,\lambda)$ should satisfy the equation (2.4).

In this case, we have four additional symmetries to those listed in Table 1. These new symmetries are related to the constants α_3, α_4, α_5 y α_6.

3. Reductions for the $\gamma_1 \neq 0$ $\gamma_2 \neq 0$ case

In what follows, we will use the following notation: p and q will be the new independent variables. $\Lambda(p,q)$, $F(p,q)$, $H(p,q)$, $N(p,q)$, $\Phi(p,q,\Lambda)$ and $\Omega(p,q,\Lambda)$ are the invariants that arise from the integration of the charasteristic system (2.3). They correspond to the integrations in λ, u, ω, m, ψ and χ. These notation can be summarized as follows:

$$\begin{aligned} x,y,t &\to p,q \\ \lambda(y,t) &\to \Lambda(p,q) \\ u(x,y,t) &\to F(p,q) \\ \omega(x,y,t) &\to H(p,q) \\ \psi(x,y,t,\lambda) &\to \Omega(p,q,\Lambda) \\ \chi(x,y,t,\lambda) &\to \Phi(p,q,\Lambda) \end{aligned} \tag{3.1}$$

According to Table 1, we have six different reductions corresponding to two arbitrary constants and four arbitrary functions. Nevertheless, the only symmetries that yield nontrivial reductions, are those related to $K_1(t)$, α_1 and α_2. Therefore, we are only considering the reductions associated to these three cases. In what follows, we consider different subcases depending which functions or constants are different or equal to zero. For instance: "$K_1 = 1$" means that we are setting all functions or constants to zero with the exception of K_1.

3.1. $K_1 = 1$

We have to solve the characteristic system (2.3) that yields the following reduced variables

$$\begin{aligned} &p = y, && q = t, \\ &u(x,y,t) = F(p,q), && \omega(x,y,t) = H(p,q), \\ &m(x,y,t) = N(p,q), && \lambda(y,t) = \Lambda(p,q), \\ &\psi(x,y,t,\lambda) = \Phi(p,q,\Lambda), && \chi(x,y,t,\lambda) = \Omega(p,q,\Lambda). \end{aligned} \tag{3.2}$$

Nevertheless, it is not an interesting reduction because the reduced equations can be easily integrated providing a trivial solution

$$\begin{aligned} F(p,q) &= b_0 e^{iZ(p,q)}, \\ H(p,q) &= b_0 e^{-iZ(p,q)}, \\ N(p,q) &= -3\gamma_1 b_0^4 p + \frac{1}{2}\int Z(p,q)_q dp. \end{aligned} \tag{3.3}$$

where $Z(p,q)$ is an arbitrary function.

3.2. $\alpha_1 = 1$

We have to solve (2.3) in order to have the reduction

$$\begin{aligned} &p = x, && q = t, \\ &u(x,y,t) = F(p,q), && \omega(x,y,t) = H(p,q), \\ &m(x,y,t) = N(p,q), && \lambda(y,t) = \Lambda(q), \\ &\psi(x,y,t,\lambda) = \Phi(p,q,\Lambda), && \chi(x,y,t,\lambda) = \Omega(p,q,\Lambda). \end{aligned} \tag{3.4}$$

By introducing these new variables into (1.2), we obtain the following reduced spectral problem

$$\begin{aligned} &\Phi_p + i\Lambda\Phi + F\Omega = 0, \\ &\Omega_p - i\Lambda\Omega + H\Phi = 0, \\ &\Phi_q + \gamma_1\{[i(-8\Lambda^4 - 4\Lambda^2 FH - 3F^2H^2 + HF_{pp} + FH_{pp} - F_pH_p) + 2\Lambda(F_pH - H_pF)]\Phi \\ &\quad +[i(F_{ppp} - 6FHF_p - 4\Lambda^2 F_p) + 2\Lambda[F_{pp} - 2F^2H - 4\Lambda^2 F]\Omega\} \\ &\quad +\gamma_2\{[2\Lambda i(2\Lambda^2 + FH) - (HF_p - FH_p)]\Phi + [2\Lambda iF_p - F_{pp} + 2F^2H + 4\Lambda^2 F]\Omega\} = 0, \\ &\Omega_q + \gamma_1\{[-i(-8\Lambda^4 - 4\Lambda^2 FH - 3F^2H^2 + FH_{pp} + HF_{pp} - F_pH_p) + 2\Lambda(H_pF - F_pH)]\Omega \\ &\quad +[-i(H_{ppp} - 6FHH_p - 4\Lambda^2 H_p) + 2\Lambda[H_{pp} - 2H^2F - 4\Lambda^2 H]\Phi\} \\ &\quad +\gamma_2\{[-2\Lambda i(2\Lambda^2 + FH) - (HF_p - FH_p)]\Omega + [-2\Lambda iH_p - H_{pp} + 2H^2F + 4\Lambda^2 H]\Phi\} = 0. \end{aligned} \tag{3.5}$$

whose compatibility requires $\Lambda = constant$ and provides the reduced equations

$$\begin{aligned}
&\left(8FHF_{pp}+6HF_p{}^2+2F^2H_{pp}+4FF_pH_p-F_{pppp}-6F^3H^2\right)\gamma_1,\\
&\quad +i\left(6FHF_p-F_{ppp}\right)\gamma_2-iF_q=0\\
&\left(8FHH_{pp}+6FH_p{}^2+2H^2F_{pp}+4HF_pH_p-H_{pppp}-6F^2H^3\right)\gamma_1.\\
&\quad -i\left(6FHH_p-H_{ppp}\right)\gamma_2+iH_q=0
\end{aligned} \tag{3.6}$$

This reduction yields the equations and the isospectral Lax pair of reference [1].

3.3. $\alpha_2 \neq 0$

The solution of the characteristic system (2.3) trivially yields the reductions

$$\begin{aligned}
&p=x, && q=y,\\
&u(x,y,t)=F(p,q), && \omega(x,y,t)=H(p,q),\\
&m(x,y,t)=N(p,q), && \lambda(y,t)=\Lambda(q),\\
&\psi(x,y,t,\lambda)=\Phi(p,q,\Lambda), && \chi(x,y,t,\lambda)=\Omega(p,q,\Lambda).
\end{aligned} \tag{3.7}$$

By inserting these new variables into (1.2), we obtain the spectral problem

$$\begin{aligned}
&\Phi_p+i\Lambda\Phi+F\Omega=0,\\
&\Omega_p-i\Lambda\Omega+H\Phi=0,\\
&2\Lambda\Phi_q=iF_q\Omega-iN_q\Phi+\gamma_1\{[i(-8\Lambda^4-4\Lambda^2FH-3F^2H^2+HF_{pp}+FH_{pp}-F_pH_p)\\
&\quad +2\Lambda(F_pH-H_pF)]\Phi+[i(F_{ppp}-6FHF_p-4\Lambda^2F_p)+2\Lambda[F_{pp}-2F^2H-4\Lambda^2F]\Omega\}\\
&\quad +\gamma_2\{[2\Lambda i(2\Lambda^2+FH)-(HF_p-FH_p)]\Phi+[2\Lambda iF_p-F_{pp}+2F^2H+4\Lambda^2F]\Omega\},\\
&2\Lambda\Omega_q=-iH_q\Phi+iN_q\Omega+\gamma_1\{[-i(-8\Lambda^4-4\Lambda^2FH-3F^2H^2+FH_{pp}+HF_{pp}-F_pH_p)\\
&\quad +2\Lambda(H_pF-F_pH)]\Omega+[-i(H_{ppp}-6FHH_p-4\Lambda^2H_p)+2\Lambda[H_{pp}-2H^2F-4\Lambda^2H]\Phi\}\\
&\quad +\gamma_2\{[-2\Lambda i(2\Lambda^2+FH)+(HF_p-FH_p)]\Omega+[-2\Lambda iH_p-H_{pp}+2H^2F+4\Lambda^2H]\Phi\}.
\end{aligned} \tag{3.8}$$

The compatibility of (3.8) implies $\Lambda = constant$ and yields the reduced equations

$$\begin{aligned}
&\left(8FHF_{pp}+6HF_p{}^2+2F^2H_{pp}+4FF_pH_p-F_{pppp}-6F^3H^2\right)\gamma_1\\
&\quad +i\left(6FHF_p-F_{ppp}\right)\gamma_2-2FN_q-F_{pq}=0,\\
&\left(8FHH_{pp}+6FH_p{}^2+2H^2F_{pp}+4HF_pH_p-H_{pppp}-6F^2H^3\right)\gamma_1\\
&\quad -i\left(6FHH_p-H_{ppp}\right)\gamma_2-2HN_q-H_{pq}=0,\\
&FH_q+HF_q+N_{pq}=0.
\end{aligned} \tag{3.9}$$

4. Reductions for the $\gamma_1 \neq 0$ $\gamma_2 = 0$ case

For this section we will set $\gamma_2 = 0$ in (1.2). According to Table 2, we have four non trivial reductions related to $K_1(t)$, α_1, α_2, and α_3, but only the last one gives us a new result. The other three reductions provides us the same results as in previous section, by imposing $\gamma_2 = 0$ in equations (3.3), (3.5) and (3.8).

4.1. $\alpha_3 = 1$

The solution for the characteristic equation (2.3) provides the following reduction

$$\begin{aligned} p &= t^{-\frac{1}{4}}x, & q &= \frac{t^{\frac{3}{4}}}{y}, \\ u(x,y,t) &= t^{-\frac{1}{4}}F(p,q), & \omega(x,y,t) &= t^{-\frac{1}{4}}H(p,q), \\ m(x,y,t) &= t^{-\frac{1}{4}}N(p,q), & \lambda(y,t) &= t^{-\frac{1}{4}}\Lambda(q), \\ \psi(x,y,t,\lambda) &= \Phi(p,q,\Lambda), & \chi(x,y,t,\lambda) &= \Omega(p,q,\Lambda). \end{aligned} \tag{4.1}$$

By inserting these new variables in (1.2) with $\gamma_2 = 0$, we obtain the following **non-isospectral** Lax pair

$$\begin{aligned} &\Phi_p + i\Lambda\Phi + F\Omega = 0, \\ &\Omega_p - i\Lambda\Omega + H\Phi = 0, \\ &\left(\frac{8\Lambda q+3}{4}\right)q\Phi_q + i\left(\frac{\Lambda p}{4} + q^2N_q\right)\Phi + q^2\Lambda_q\Phi + \left(\frac{pF}{4} - iq^2F_q\right)\Omega \\ &\quad +\gamma_1[\{2\Lambda(HF_p - FH_p) + i(-8\Lambda^4 - 4\Lambda^2FH + HF_{pp} + HF_{pp} - F_pH_p - 3F^2H^2)\}\Phi \\ &\quad +\{2\Lambda F_{pp} - 4\Lambda F^2H - 8\Lambda^3F + i(F_{ppp} - 4\Lambda^2F_p - 6FHF_p)\}\Omega] = 0, \\ &\left(\frac{8\Lambda q+3}{4}\right)q\Omega_q - i\left(\frac{\Lambda p}{4} + q^2N_q\right)\Omega + q^2\Lambda_q\Omega + \left(\frac{Hp}{4} + iq^2H_q\right)\Phi \\ &\quad +\gamma_1[\{2\Lambda(FH_p - HF_p) - i(-8\Lambda^4 - 4\Lambda^2FH + FH_{pp} + FH_{pp} - F_pH_p - 3F^2H^2)\}\Omega \\ &\quad +\{2\Lambda H_{pp} - 4\Lambda H^2F - 8\Lambda^3H - i(H_{ppp} - 4\Lambda^2H_p - 6FHH_p)\}\Phi] = 0. \end{aligned} \tag{4.2}$$

The compatibility of the above equations yields the reduced equations

$$\begin{aligned} &\left(-6F^3H^2 + 2F^2H_{pp} + 8FHF_{pp} + 4FF_pH_p + 6HF_p{}^2 - F_{pppp}\right)\gamma_1 \\ &\quad +2q^2FN_q + q^2F_{pq} + \frac{i}{4}(pF_p - 3qF_q + F) = 0, \\ &\left(-6F^2H^3 + 2H^2F_{pp} + 8FHH_{pp} + 4HF_pH_p + 6FH_p{}^2 - H_{pppp}\right)\gamma_1 \\ &\quad +2q^2HN_q + q^2H_{pq} - \frac{i}{4}(pH_p - 3qH_q + H) = 0, \\ &FH_q + HF_q + N_{pq} = 0, \end{aligned} \tag{4.3}$$

and the **non-isospectral** condition

$$\Lambda_q = \frac{\Lambda}{q(8\Lambda q + 3)}. \tag{4.4}$$

5. Reductions for the $\gamma_1 = 0$, $\gamma_2 \neq 0$ case

In this section we will set $\gamma_1 = 0$ in (1.2). From Table 3, we can realize that, as in the previous section, we have four non trivial reductions related to $K_1(t)$, α_1, α_2, and α_3. The first three reductions provides the same results of section 3 by setting $\gamma_1 = 0$.

5.1. $\alpha_3 = 1$

The characteristic equation (2.3) yields the reduction

$$
\begin{aligned}
&p = t^{-\frac{1}{3}}x, && q = \frac{t^{\frac{2}{3}}}{y},\\
&u(x,y,t) = t^{-\frac{1}{3}}F(p,q), && \omega(x,y,t) = t^{-\frac{1}{3}}H(p,q),\\
&m(x,y,t) = t^{-\frac{1}{3}}N(p,q), && \lambda(y,t) = t^{-\frac{1}{3}}\Lambda(q),\\
&\psi(x,y,t,\lambda) = \Phi(p,q,\Lambda), && \chi(x,y,t,\lambda) = \Omega(p,q,\Lambda).
\end{aligned}
\tag{5.1}
$$

The introduction of these reductions in the Lax pair (1.2) yields the following **non-isospectral** linear problem

$$
\begin{aligned}
&\Phi_p + i\Lambda\Phi + F\Omega = 0,\\
&\Omega_p - i\Lambda\Omega + H\Phi = 0,\\
&\left(\frac{2}{3}q + 2\Lambda q^2\right)\Phi_q + \frac{p}{3}\left(F\Omega + i\Lambda\Phi\right) + q^2\left(\Lambda_q\Phi + iN_q\Phi - iF_q\Omega\right)\\
&\quad +[(2iFH\Lambda + 4i\Lambda^3 + FH_p - HF_p)\Phi + (2HF^2 + 4F\Lambda^2 + 2i\Lambda F_p - F_{pp})\Omega]\gamma_2 = 0,\\
&\left(\frac{2}{3}q + 2\Lambda q^2\right)\Omega_q + \frac{p}{3}\left(H\Phi - i\Lambda\Omega\right) + q^2\left(\Lambda_q\Omega - iN_q\Omega + iH_q\Phi\right)\\
&\quad +[(-2iFH\Lambda - 4i\Lambda^3 - FH_p + HF_p)\Omega + (2FH^2 + 4H\Lambda^2 - 2i\Lambda H_p - H_{pp})\Phi]\gamma_2 = 0.
\end{aligned}
\tag{5.2}
$$

The compatibility of the above system provides the following equations

$$
\begin{aligned}
&(6FHF_p - F_{ppp})\,i\gamma_2 + \frac{i}{3}(F + pF_p - 2qF_q) + q^2(F_{pq} + FN_q) = 0,\\
&-(6FHH_p - H_{ppp})\,i\gamma_2 - \frac{i}{3}(H + pH_p - 2qH_q) + q^2(H_{pq} + HN_q) = 0,\\
&FH_q + HF_q + N_{pq} = 0,
\end{aligned}
\tag{5.3}
$$

and the **non-isospectral** condition

$$
\Lambda_q = \frac{1}{2}\,\frac{\Lambda}{q(3\Lambda q + 1)}. \tag{5.4}
$$

6. Reductions for the $\gamma_1 = 0$, $\gamma_2 = 0$ case

For this section we will set $\gamma_1 = 0$ and $\gamma_2 = 0$ in (1.2). We new four additional non trivial reductions related to the constants α_3, α_4, α_5 y α_6.

6.1. $\alpha_3 = 1$

Characteristic equation (2.3) yields the reductions

$$\begin{aligned} p &= \frac{x}{t}, & q &= y, \\ u(x,y,t) &= \frac{F(p,q)}{t}, & \omega(x,y,t) &= \frac{H(p,q)}{t}, \\ m(x,y,t) &= \frac{N(p,q)}{t}, & \lambda(y,t) &= \frac{\Lambda(q)}{t}, \\ \psi(x,y,t,\lambda) &= \Phi(p,q,\Lambda), & \chi(x,y,t,\lambda) &= \Omega(p,q,\Lambda). \end{aligned} \tag{6.1}$$

By inserting these new variables into (1.2), where we have taken previously $\gamma_1 = \gamma_2 = 0$, we obtain

$$\begin{aligned} &\Phi_p + i\Lambda\Phi + F\Omega = 0, \\ &\Omega_p - i\Lambda\Omega + H\Phi = 0, \\ &2\Lambda\Phi_q = \left(\frac{1}{2} + ip\Lambda - iN_q\right)\Phi + (pF + iF_q)\Omega, \\ &2\Lambda\Omega_q = \left(\frac{1}{2} - ip\Lambda + iN_q\right)\Omega + (pH - iH_q)\Phi, \end{aligned} \tag{6.2}$$

whose compatibility provides us the :

$$\Lambda_q = -\frac{1}{2}$$

and the reduced equations

$$\begin{aligned} &2FN_q + F_{pq} - ipF_p - iF = 0, \\ &2HN_q + H_{pq} + ipH_p + iH = 0, \\ &FH_q + HF_q + N_{pq} = 0. \end{aligned} \tag{6.3}$$

6.2. $\alpha_4 = 1$

The reductions obtained through the integration of (2.3) are:

$$\begin{aligned} p &= x, & q &= \frac{t}{y}, \\ u &= F(p,q), & w &= H(p,q), \\ m &= N(p,q), & \lambda &= \Lambda(q), \\ \psi &= \Phi(p,q), & \chi &= \Omega(p,q). \end{aligned} \tag{6.4}$$

The Lax pair (1.2) reduces to the spectral problem

$$\begin{aligned}
&\Phi_p + i\Lambda\Phi + F\Omega = 0,\\
&\Omega_p - i\Lambda\Omega + H\Phi = 0,\\
&\Phi_q - iq\Omega F_q + iq\Phi N_q + 2\Lambda q\Phi_q = 0,\\
&\Omega_q + iq\Phi H_q - iq\Omega N_q + 2\Lambda q\Omega_q = 0.
\end{aligned} \tag{6.5}$$

whose compatibility provides us: $\Lambda = constant$ and the reduced equations

$$\begin{aligned}
&-iF_q + 2qFN_q + qF_{pq} = 0,\\
&iH_q + 2qHN_q + qH_{pq} = 0,\\
&H_qF + HF_q + N_{pq} = 0.
\end{aligned} \tag{6.6}$$

6.3. $\alpha_5 = 1$

The integration of the characteristic system (2.3) yields the reduction

$$\begin{aligned}
&p = \frac{x}{t}, && q = \frac{y}{t},\\
&u = \frac{F(p,q)}{t}\exp(itpq),, && w = \frac{H(p,q)}{t}\exp(-itpq),\\
&m = \frac{N(p,q)}{t}, && \lambda = \frac{-qt + \Lambda(p,q)}{2t},\\
&\psi = \frac{\Phi(p,q)}{\sqrt{t}}\exp\left(\frac{itpq}{2}\right), && \chi = \frac{\omega(p,q)}{\sqrt{t}}\exp\left(\frac{-itpq}{2}\right).
\end{aligned} \tag{6.7}$$

The reduced Lax pair is in this case

$$\begin{aligned}
&\Phi_p + \frac{1}{2}i\Lambda\Phi + F\Omega = 0,\\
&\Omega_p - \frac{1}{2}i\Lambda\Omega + H\Phi = 0,\\
&\Lambda\Phi_q - iF_q\Omega + iN_q\Phi = 0,\\
&\Lambda\Omega_q + iH_q\Phi - iN_q\Omega = 0.
\end{aligned} \tag{6.8}$$

whose compatibility implies that $\Lambda = constant$ and the reduced equations

$$\begin{aligned}
&2FN_q + F_{pq} = 0,\\
&2HN_q + H_{pq} = 0,\\
&FH_q + HF_q + N_{pq} = 0.
\end{aligned} \tag{6.9}$$

6.4. $\alpha_6 = 1$

These are the reductions obtained through the integration of (2.3)

$$\begin{aligned} p &= x, & q &= t, \\ u &= F(p,q)\exp\left(i\frac{p}{q}y\right), & \omega &= H(p,q)\exp\left(-i\frac{p}{q}y\right), \\ m &= N(p,q), & \lambda &= \frac{2\Lambda(p,q)-y}{2q}, \\ \psi &= \Phi(p,q)\exp\left(i\frac{p}{2q}y\right), & \chi &= \Omega(p,q)\exp\left(-i\frac{p}{2q}y\right), \end{aligned} \tag{6.10}$$

The reduced Lax pair is

$$\begin{aligned} &q\Phi_p + qF\Omega + i\Lambda\Phi = 0, \\ &q\Omega_p + qH\Phi - i\Lambda\Omega = 0, \\ &-2q^2\Phi_q + 2pqF\Omega + 2ip\Lambda\Phi - q\Phi = 0, \\ &-2q^2\Omega_q + 2pqH\Phi - 2ip\Lambda\Omega - q\Omega = 0, \end{aligned} \tag{6.11}$$

whose compatibility implies $\Lambda = constant$. The reduced equations are

$$\begin{aligned} &pF_p + qF_q + F = 0, \\ &pH_p + qH_q + H = 0. \end{aligned} \tag{6.12}$$

7. Conclusions

We have determined the classical Lie symmetries of a rather complicated non-isospectral Lax pair in $2+1$ dimensions, which generalizes the well known nonlinear Schrödinger equation. This procedure allows us to get the infinitesimals related to the independent variables and fields, as well as those associated to the spectral parameter and eigenfunctions. Four different sets of symmetries can be obtained depending whether the parameters γ_1, γ_2 are zero or different from zero.

The next step is the identification of the reductions associated to each symmetry. Our procedure has the advantage that the reduced spectral parameter and eigenfunctions are simultaneously determined. Therefore, we can have, not only the reduced equations, but the reduced spectral problem. Actually, three special cases have been obtained, in which the reduced spectral problem in $1+1$ dimensions is yet non-isospectral.

We should remark that the reduced equations are, in most of the cases, quite complicated and, in many cases, non-autonomous. The identification of the integrability of these equations and its associated Lax pair could be in general a non easy problem. Nevertheless, the identification of the symmetries of (1.2) yields the right reduced spectral problem.

Acknowledgments

This research has been supported by MINECO (Grants MAT2013-46308 and MAT2016-75955) and Junta de Castilla y León (Grant SA045U16). P. Albares acknowledges a fellowship from the Junta de Castilla y Leon.

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